

From pro- p Iwahori-Hecke modules to (φ, Γ) -modules, II

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Abstract

Let \mathfrak{o} be the ring of integers in a finite extension field of \mathbb{Q}_p , let k be its residue field. Let G be a split reductive group over \mathbb{Q}_p , let $\mathcal{H}(G, I_0)$ be its pro- p -Iwahori Hecke \mathfrak{o} -algebra. In [2] we introduced a general principle how to assign to a certain additionally chosen datum $(C^{(\bullet)}, \phi, \tau)$ an exact functor $M \mapsto \mathbf{D}(\Theta_* \mathcal{V}_M)$ from finite length $\mathcal{H}(G, I_0)$ -modules to (φ^r, Γ) -modules. In the present paper we concretely work out such data $(C^{(\bullet)}, \phi, \tau)$ for the classical matrix groups. We show that the corresponding functor identifies the set of (standard) supersingular $\mathcal{H}(G, I_0) \otimes_{\mathfrak{o}} k$ -modules with the set of (φ^r, Γ) -modules satisfying a certain symmetry condition.

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1 Introduction

Let \mathfrak{o} be the ring of integers in a finite extension field of \mathbb{Q}_p , let k be its residue field. Let G be a split reductive group over \mathbb{Q}_p , let T be a maximal split torus in G , let

I_0 be a pro- p -Iwahori subgroup fixing a chamber C in the T -stable apartment of the semi simple Bruhat Tits building of G . Let $\mathcal{H}(G, I_0)$ be the pro- p -Iwahori Hecke \mathfrak{o} -algebra. Let $\text{Mod}^{\text{fin}}(\mathcal{H}(G, I_0))$ denote the category of $\mathcal{H}(G, I_0)$ -modules of finite \mathfrak{o} -length. From a certain additional datum $(C^{(\bullet)}, \phi, \tau)$ we constructed in [2] an exact functor $M \mapsto \mathbf{D}(\Theta_* \mathcal{V}_M)$ from $\text{Mod}^{\text{fin}}(\mathcal{H}(G, I_0))$ to the category of étale (φ^r, Γ) -modules (with $r \in \mathbb{N}$ depending on ϕ). For $G = \text{GL}_2(\mathbb{Q}_p)$, when precomposed with the functor of taking I_0 -invariants, this yields the functor from smooth \mathfrak{o} -torsion representations of $\text{GL}_2(\mathbb{Q}_p)$ (or at least from those generated by their I_0 -invariants) to étale (φ, Γ) -modules which plays a crucial role in Colmez' construction of a p -adic local Langlands correspondence for $\text{GL}_2(\mathbb{Q}_p)$. In [2] we studied in detail the functor $M \mapsto \mathbf{D}(\Theta_* \mathcal{V}_M)$ when $G = \text{GL}_{d+1}(\mathbb{Q}_p)$ for $d \geq 1$. In [6] the situation has been analysed for $G = \text{SL}_{d+1}(\mathbb{Q}_p)$. The purpose of the present paper is to explain how the general construction of [2] can be installed concretely for other classical matrix groups G (as well as for G 's of type E_6, E_7).

Recall that $C^{(\bullet)} = (C = C^{(0)}, C^{(1)}, C^{(2)}, \dots)$ is a minimal gallery, starting at C , in the T -stable apartment, that $\phi \in N(T)$ is a 'period' of $C^{(\bullet)}$ and that τ is a homomorphism from \mathbb{Z}_p^\times to T , compatible with ϕ in a suitable sense. The above $r \in \mathbb{N}$ is just the length of ϕ . It turns out that τ must be a minuscule fundamental coweight (at least if the underlying root system is simple). Conversely, any minuscule fundamental coweight τ can be included into a datum $(C^{(\bullet)}, \phi, \tau)$, in such a way that some power of τ is a power of ϕ .

For $G = \text{GL}_{d+1}(\mathbb{Q}_p)$ we gave explicit choices of $(C^{(\bullet)}, \phi, \tau)$ with $r = 1$ in [2] (there are essentially just two choices, and these are 'dual' to each other). In the present paper we work out 'privileged' choices $(C^{(\bullet)}, \phi, \tau)$ for the classical matrix groups, as well as for G 's of type E_6, E_7 . We mostly consider G with connected center Z . Our choices of $(C^{(\bullet)}, \phi, \tau)$ are such that $\phi \in N(T)$ projects (modulo ZT_0 , where T_0 denotes the maximal bounded subgroup of T) to the affine Weyl group (viewed as a subgroup of $N(T)/ZT_0$). In particular, up to modifications by elements of Z these ϕ can also be included into data $(C^{(\bullet)}, \phi, \tau)$ for the other G 's with the same underlying root system, not necessarily with connected center. We indicate these modifications along the way. Notice that the $\phi \in N(T)$ considered in [2] for $G = \text{GL}_{d+1}(\mathbb{Q}_p)$ does *not* project to the affine Weyl group, only its $(d+1)$ -st power (which is considered here) does so. But since the discussion is essentially the same, our treatment of the case A here is very brief.

In either case we work out the behaviour of the functor $M \mapsto \mathbf{D}(\Theta_* \mathcal{V}_M)$ on those $\mathcal{H}(G, I_0)_k = \mathcal{H}(G, I_0) \otimes_{\mathfrak{o}} k$ -modules which we call 'standard supersingular'. Roughly speaking, these are induced from characters of the pro- p -Iwahori Hecke algebra of the corresponding simply connected group. Each irreducible supersingular $\mathcal{H}(G, I_0)_k$ -module is contained in (and in 'most' cases is equal to) a standard supersingular $\mathcal{H}(G, I_0)_k$ -module

(and the very few standard supersingular $\mathcal{H}(G, I_0)_k$ -modules which are not irreducible supersingular are easily identified). We show that our functor induces a bijection between the set of isomorphism classes (resp. certain packets of such if $G = \mathrm{SO}_{2d+1}(\mathbb{Q}_p)$) of standard supersingular $\mathcal{H}(G, I_0)_k$ -modules and the set of (isomorphism classes of) étale (φ^r, Γ) -modules over $k_{\mathcal{E}} = k((t))$ which satisfy a certain symmetry condition (depending on the root system underlying G). They are direct sums of one dimensional étale (φ^r, Γ) -modules, their dimension is the k -dimension of the corresponding standard supersingular $\mathcal{H}(G, I_0)_k$ -module.

The interest in étale (φ^r, Γ) -modules lies in their relation with $\mathrm{Gal}_{\mathbb{Q}_p}$ -representations. For any $r \in \mathbb{N}$ there is an exact functor from the category of étale (φ^r, Γ) -modules to the category of étale (φ, Γ) -modules (it multiplies the rank by the factor r), and by means of Fontaine's functor, the latter one is equivalent with the category of $\mathrm{Gal}_{\mathbb{Q}_p}$ -representations.

In [2] we also explained that a datum $(C^{(\bullet)}, \phi)$ alone, i.e. without a τ as above, can be used to define an exact functor $M \mapsto \mathbf{D}(\Theta_* \mathcal{V}_M)$ from $\mathrm{Mod}^{\mathrm{fin}}(\mathcal{H}(G, I_0))$ to the category of étale (φ^r, Γ_0) -modules, where Γ_0 denotes the maximal pro- p -subgroup of $\Gamma \cong \mathbb{Z}_p^\times$. Such data $(C^{(\bullet)}, \phi)$ are not tied to co minuscule coweights and exist in abundance. We do not discuss them here.

We hope that, beyond its immediate purposes as described above, the present paper may also be a useful reference for explicit descriptions of the pro- p -Iwahori algebra (in particular with respect to the various Weyl groups involved) of classical matrix groups other than GL_d (we could not find such descriptions in the literature).

The outline is as follows. In section 2 we explain the functor from étale (φ^r, Γ) -modules to étale (φ, Γ) -modules, and we introduce the 'symmetric' étale (φ^r, Γ) -modules mentioned above, for each of the root systems C , B , D and A . In section 3, Lemma 3.1, we discuss the relation between the data $(C^{(\bullet)}, \phi, \tau)$ and minuscule fundamental coweights. Our discussions of classical matrix groups G in section 4 are just concrete incarnations of Lemma 3.1, although in neither of these cases there is a need to make formal reference to Lemma 3.1. On the other hand, in our discussion of the cases E_6 and E_7 in section 5 we do invoke Lemma 3.1. We tried to synchronize our discussions of the various matrix groups. As a consequence, arguments repeat themselves, and we do not write them out again and again. In the appendix we record calculations relevant for the cases E_6 and E_7 , carried out with the help of the computer algebra system *sage*.

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2 (φ^r, Γ) -modules

We often regard elements of \mathbb{F}_p^\times as elements of \mathbb{Z}_p^\times by means of the Teichmüller lifting. In $\mathrm{GL}_2(\mathbb{Z}_p)$ we define the subgroups

$$\Gamma = \begin{pmatrix} \mathbb{Z}_p^\times & 0 \\ 0 & 1 \end{pmatrix}, \quad \Gamma_0 = \begin{pmatrix} 1 + p\mathbb{Z}_p & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathfrak{N}_0 = \begin{pmatrix} 1 & \mathbb{Z}_p \\ 0 & 1 \end{pmatrix}$$

and the elements

$$\varphi = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, \quad \nu = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad h(x) = \begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}, \quad \gamma(x) = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$$

where $x \in \mathbb{Z}_p^\times$. Let $\mathcal{O}_\mathcal{E}^+ = \mathfrak{o}[[\mathfrak{N}_0]]$ denote the completed group ring of \mathfrak{N}_0 over \mathfrak{o} . Let $\mathcal{O}_\mathcal{E}$ denote the p -adic completion of the localization of $\mathcal{O}_\mathcal{E}^+$ with respect to the complement of $\pi_K \mathcal{O}_\mathcal{E}^+$, where $\pi_K \in \mathfrak{o}$ is a uniformizer. In the completed group ring $k_\mathcal{E}^+ = k[[\mathfrak{N}_0]]$ we put $t = [\nu] - 1$. Let $k_\mathcal{E} = \mathrm{Frac}(k_\mathcal{E}^+) = \mathcal{O}_\mathcal{E} \otimes_\mathfrak{o} k$. We identify $k_\mathcal{E}^+ = k[[t]]$ and $k_\mathcal{E} = k((t))$. For definitions and notational conventions concerning étale φ^r -modules and étale (φ^r, Γ) -modules we refer to [2].

Let $r \in \mathbb{N}$. Let $\mathbf{D} = (\mathbf{D}, \varphi_\mathbf{D}^r)$ be an étale φ^r -module over $\mathcal{O}_\mathcal{E}$. For $0 \leq i \leq r-1$ let $\mathbf{D}^{(i)} = \mathbf{D}$ be a copy of \mathbf{D} . For $1 \leq i \leq r-1$ define $\varphi_\mathbf{D} : \mathbf{D}^{(i)} \rightarrow \mathbf{D}^{(i-1)}$ to be the identity map on \mathbf{D} , and define $\varphi_\mathbf{D} : \mathbf{D}^{(0)} \rightarrow \mathbf{D}^{(r-1)}$ to be the structure map $\varphi_\mathbf{D}^r$ on \mathbf{D} . Together we obtain a \mathbb{Z}_p -linear endomorphism $\varphi_\mathbf{D}$ on

$$\tilde{\mathbf{D}} = \bigoplus_{i=0}^{r-1} \mathbf{D}^{(i)}.$$

Define an $\mathcal{O}_\mathcal{E}$ -action on $\tilde{\mathbf{D}}$ by the formula $x \cdot ((d_i)_{0 \leq i \leq r-1}) = (\varphi_{\mathcal{O}_\mathcal{E}}^i(x) d_i)_{0 \leq i \leq r-1}$. Then the endomorphism $\varphi_\mathbf{D}$ of $\tilde{\mathbf{D}}$ is semilinear with respect to this $\mathcal{O}_\mathcal{E}$ -action, hence it defines on $\tilde{\mathbf{D}}$ the structure of an étale φ -module over $\mathcal{O}_\mathcal{E}$, see [2], section 6.3.

Let Γ' be an open subgroup of Γ , let \mathbf{D} be an étale (φ^r, Γ') -module over $\mathcal{O}_\mathcal{E}$. Define an action of Γ' on $\tilde{\mathbf{D}}$ by

$$\gamma \cdot ((d_i)_{0 \leq i \leq r-1}) = (\gamma \cdot d_i)_{0 \leq i \leq r-1}.$$

Lemma 2.1. *The Γ' -action on $\tilde{\mathbf{D}}$ commutes with $\varphi_\mathbf{D}$ and is semilinear with respect to the $\mathcal{O}_\mathcal{E}$ -action, hence we obtain on $\tilde{\mathbf{D}}$ the structure of an étale (φ, Γ') -module over $\mathcal{O}_\mathcal{E}$. We thus obtain an exact functor from the category of étale (φ^r, Γ') -modules to the category of étale (φ, Γ') -modules over $\mathcal{O}_\mathcal{E}$.*

PROOF: This is immediate from the respective properties of the Γ' -action on \mathbf{D} . \square

The conjugation action of Γ on \mathfrak{N}_0 allows us to define for each $x \in \mathbb{Z}_p^\times$ an automorphism $f \mapsto \gamma(x)f\gamma(x^{-1})$ of $k[[t]] = k_\mathcal{E}^+$.

Lemma 2.2. For $x \in \mathbb{F}_p^\times$ and $m, n \in \mathbb{Z}_{\geq 0}$ we have

$$(1) \quad \gamma(x)t^{np^{r^m}}\gamma(x^{-1}) - (xt)^{np^{r^m}} \in t^{(n+1)p^{r^m}}k[[t]].$$

PROOF: Denote again by x be the representative of x in $[1, p-1]$. We compute $[\nu]^x - 1 = \sum_{j=1}^x \binom{x}{j}([\nu] - 1)^j = \sum_{j=1}^x \binom{x}{j}t^j$, hence

$$\begin{aligned} \gamma(x)t\gamma(x^{-1}) - xt &= ([\nu]^x - 1) - xt \in t^2k[[t]], \\ \gamma(x)t^n\gamma(x^{-1}) - (xt)^n &\in t^{n+1}k[[t]], \\ \gamma(x)t^{np^{r^m}}\gamma(x^{-1}) - (xt)^{np^{r^m}} &= (\gamma(x)t^n\gamma(x^{-1}) - (xt)^n)^{p^{r^m}} \in t^{(n+1)p^{r^m}}k[[t]]. \end{aligned}$$

□

Lemma 2.3. (a) Let \mathbf{D} be a one-dimensional étale (φ^r, Γ) -module over $k_{\mathcal{E}}$. There exists a basis element g for \mathbf{D} , uniquely determined integers $0 \leq s(\mathbf{D}) \leq p-2$ and $1 \leq n(\mathbf{D}) \leq p^r - 1$ and a uniquely determined scalar $\xi(\mathbf{D}) \in k^\times$ such that

$$\begin{aligned} \varphi_{\mathbf{D}}^r g &= \xi(\mathbf{D})t^{n(\mathbf{D})+1-p^r}g \\ \gamma(x)g - x^{s(\mathbf{D})}g &\in t \cdot k_{\mathcal{E}}^+ \cdot g \end{aligned}$$

for all $x \in \mathbb{Z}_p^\times$. Thus, one may define $0 \leq k_i(\mathbf{D}) \leq p-1$ by $n(\mathbf{D}) = \sum_{i=0}^{r-1} k_i(\mathbf{D})p^i$. One has $n(\mathbf{D}) \equiv 0$ modulo $(p-1)$.

(b) For any given integers $0 \leq s \leq p-2$ and $1 \leq n \leq p^r - 1$ with $n \equiv 0$ modulo $(p-1)$ and any scalar $\xi \in k^\times$ there is a uniquely determined (up to isomorphism) one-dimensional étale (φ^r, Γ) -module \mathbf{D} over $k_{\mathcal{E}}$ with $s = s(\mathbf{D})$ and $n = n(\mathbf{D})$ and $\xi = \xi(\mathbf{D})$.

PROOF: (a) Begin with an arbitrary basis element g_0 for \mathbf{D} ; then $\varphi_{\mathbf{D}}^r g_0 = Fg_0$ for some unit $F \in k((t)) = k_{\mathcal{E}}$. After multiplying g_0 with a suitable power of t we may assume $F = \xi t^m(1 + t^{n_0}F_0)$ for some $0 \geq m \geq 2 - p^r$, some $\xi \in k^\times$, some $n_0 > 0$, and some $F_0 \in k[[t]]$ (use $t^{p^r}\varphi^r = \varphi^r t$). For $g_1 = (1 + t^{n_0}F_0)g_0$ we then get $\varphi_{\mathbf{D}}^r g_1 = \xi t^m(1 + t^{n_1}F_1)g_1$ for some $n_1 > n_0 > 0$ and some $F_1 \in k[[t]]$. We put $g_2 = (1 + t^{n_1}F_1)g_1$. We may continue in this way. In the limit we get, by completeness, $g = g_\infty \in \mathbf{D}$ such that $\varphi_{\mathbf{D}}^r g = \xi t^m g$. It is clear that $\xi(\mathbf{D}) = \xi$ and $n(\mathbf{D}) = p^r - 1 + m$ are well defined.

Next, as $\gamma(x)t^{p^r-1}$ (for $x \in \mathbb{Z}_p^\times$) is topologically nilpotent (in the topological group Γ) and acts by an automorphism on \mathbf{D} , there is some unit $F_x \in k[[t]]$ with $\gamma(x)g = F_x g$. Thus, the action of Γ on \mathbf{D} induces an action of Γ on $k[[t]]/tk[[t]] \cong k$, and this action is given by a homomorphism $\mathbb{Z}_p^\times \cong \Gamma \rightarrow k^\times$, $x \mapsto x^{s(\mathbf{D})}$ for some $0 \leq s(\mathbf{D}) \leq p-2$. Computing modulo $t \cdot k_{\mathcal{E}}^+ \cdot g$ we have (with $n = n(\mathbf{D})$, $s = s(\mathbf{D})$, $\xi = \xi(\mathbf{D})$)

$$x^s g \equiv \gamma(x)g = \xi^{-1}\gamma(x)t^{p^r-n-1}\varphi_{\mathbf{D}}^r g = \xi^{-1}(\gamma(x)t^{p^r-n-1}\gamma(x^{-1}))\gamma(x)\varphi_{\mathbf{D}}^r g$$

$$\stackrel{(i)}{\equiv} \xi^{-1}(xt)^{p^r-n-1}\gamma(x)\varphi_{\mathbf{D}}^r g \equiv x^s \xi^{-1}(xt)^{p^r-n-1}\varphi_{\mathbf{D}}^r g = x^{s+p^r-n-1}g.$$

Here (i) follows from $\gamma(x)t^{p^r-n-1}\gamma(x^{-1}) - (xt)^{p^r-n-1} \in t^{p^r-n}k[[t]]$, formula (1). In short, we have $g \equiv x^{p^r-n-1}g$ for all $x \in \mathbb{F}_p^\times$. We deduce $n \equiv 0$ modulo $p-1$.

(b) Put $D = k[[t]] = k_{\mathcal{E}}^+$ and $D^* = \text{Hom}_k^{\text{ct}}(k_{\mathcal{E}}^+, k)$. Endow D^* with an action of $k_{\mathcal{E}}^+$ by putting $(\alpha \cdot \ell)(x) = \ell(\alpha \cdot x)$ for $\alpha \in k_{\mathcal{E}}^+$, $\ell \in D^*$ and $x \in D$. For $j \geq 0$ define $\ell_j \in D^*$ by

$$\ell_j(\sum_{i \geq 0} a_i t^i) = a_j.$$

Then $\{\ell_j\}_{j \geq 0}$ is a k -basis of D^* , and we have $t\ell_{j+1} = \ell_j$ and $t\ell_0 = 0$. Define $\varphi^r : D^* \rightarrow D^*$ to be the k -linear map with $\varphi^r(\ell_j) = \xi^{-1}\ell_{p^r j+n}$. We then find

$$(t^{p^r}(\varphi^r(\ell_j)))(\sum_{i \geq 0} a_i t^i) = (\xi^{-1}\ell_{p^r j+n})(\sum_{i \geq 0} a_i t^{i+p^r}) = \xi^{-1}a_{p^r(j-1)+n},$$

$$(\varphi^r(t\ell_j))(\sum_{i \geq 0} a_i t^i) = \xi^{-1}\ell_{p^r(j-1)+n}(\sum_{i \geq 0} a_i t^i) = \xi^{-1}a_{p^r(j-1)+n}$$

(with $a_i = 0$ resp. $\ell_i = 0$ for $i < 0$). As $t^{p^r}\varphi^r = \varphi^r t$ in $k_{\mathcal{E}}^+[\varphi^r]$ this means that we have defined an action of $k_{\mathcal{E}}^+[\varphi^r]$ on D^* . Next, for $x \in \mathbb{F}_p^\times$ and $b, m \in \mathbb{Z}_{\geq 0}$ we put

$$\gamma(x) \cdot \ell_{-b+n \sum_{i=0}^{m-1} p^{ri}} = x^{-s}(\gamma(x)t^b\gamma(x^{-1}))\ell_{-b+n \sum_{i=0}^{m-1} p^{ri}},$$

or equivalently,

$$(2) \quad \gamma(x) \cdot t^b(\varphi^r)^m \ell_0 = x^{-s}(\gamma(x)t^b\gamma(x^{-1}))(\varphi^r)^m \ell_0.$$

We claim that this defines an action of γ on D^* . In order to check well definedness, i.e. independence on the choice of b and m when only $-b + n \sum_{i=0}^{m-1} p^{ri}$ is given, it is enough to consider the equations $t^b(\varphi^r)^m \ell_0 = \xi t^{b+np^{rm}}(\varphi^r)^{m+1} \ell_0$ and to compare the respective effect of $\gamma(x)$ (through formula (2)) on either side. Namely, we need to see

$$\xi^{-1}x^{-s}(\gamma(x)t^b\gamma(x^{-1}))(\varphi^r)^m \ell_0 = x^{-s}(\gamma(x)t^{b+np^{rm}}\gamma(x^{-1}))(\varphi^r)^{m+1} \ell_0,$$

or equivalently,

$$(\gamma(x)t^b\gamma(x^{-1}))(\xi^{-1} - (\gamma(x)t^{np^{rm}}\gamma(x^{-1}))\varphi^r)(\varphi^r)^m \ell_0 = 0.$$

For this it is enough to see that $\xi^{-1} - (\gamma(x)t^{np^{rm}}\gamma(x^{-1}))\varphi^r$ annihilates $(\varphi^r)^m \ell_0$. Now $n < p^r$ implies $(n+1)p^{rm} > n \sum_{i=0}^m p^{ri}$ and hence that $t^{(n+1)p^{rm}}$ annihilates $(\varphi^r)^{m+1} \ell_0 = \xi^{-m-1}\ell_{n \sum_{i=0}^m p^{ri}}$. Therefore it is enough, by formula (1), to show that $\xi^{-1} - (xt)^{np^{rm}}\varphi^r$ annihilates $(\varphi^r)^m \ell_0$. But $n \equiv 0$ modulo $(p-1)$ means $x^{np^{rm}} = 1$. Our claim follows since $\xi^{-1}(\varphi^r)^m \ell_0 = t^{np^{rm}}(\varphi^r)^{m+1} \ell_0$.

In fact, we have defined an action of $k_{\mathcal{E}}^+[\varphi^r, \Gamma]$ on D^* . Namely, it is clear from the definitions that the relations $\gamma(x) \cdot t = (\gamma(x)t\gamma(x^{-1})) \cdot \gamma(x)$ in $k_{\mathcal{E}}^+[\varphi^r, \Gamma]$ are respected. That the actions of $\gamma(x)$ and φ^r on D^* commute follows easily from the relations $\gamma(x)\varphi^r = \varphi^r\gamma(x)$ and $t^{p^r}\varphi^r = \varphi^rt$ in $k_{\mathcal{E}}^+[\varphi^r, \Gamma]$. Notice that

$$\gamma(x)\ell_0 = x^{-s}\ell_0 \quad \text{for all } x \in \mathbb{Z}_p^\times.$$

Now passing to the dual $D \cong (D^*)^*$ of D^* yields a non degenerate (ψ^r, Γ) -module over $k_{\mathcal{E}}^+$ with an associated étale (φ^r, Γ) -module \mathbf{D} over $k_{\mathcal{E}}$ with $s = s(\mathbf{D})$ and $n = n(\mathbf{D})$ and $\xi = \xi(\mathbf{D})$; this is explained in [2] Lemma 6.4. This dualization argument also proves the uniqueness of \mathbf{D} . \square

Definition: We say that an étale (φ^r, Γ) -module \mathbf{D} over $k_{\mathcal{E}}$ is C -symmetric if it admits a direct sum decomposition $\mathbf{D} = \mathbf{D}_1 \oplus \mathbf{D}_2$ with one-dimensional étale (φ^r, Γ) -modules $\mathbf{D}_1, \mathbf{D}_2$ satisfying the following conditions (1), (2C) and (3C):

- (1) $k_i(\mathbf{D}_1) = k_{r-1-i}(\mathbf{D}_2)$ for all $0 \leq i \leq r-1$
- (2C) $\xi(\mathbf{D}_1) = \xi(\mathbf{D}_2)$
- (3C) $s(\mathbf{D}_2) - s(\mathbf{D}_1) \equiv \sum_{i=0}^{r-1} ik_i(\mathbf{D}_1) \pmod{p-1}$
- (4) $k_{\bullet}(\mathbf{D}_1) \notin \{(0, \dots, 0), (p-1, \dots, p-1)\}$

Definition: We say that an étale (φ^r, Γ) -module \mathbf{D} over $k_{\mathcal{E}}$ is B -symmetric if r is odd and if \mathbf{D} admits a direct sum decomposition $\mathbf{D} = \mathbf{D}_1 \oplus \mathbf{D}_2$ with one-dimensional étale (φ^r, Γ) -modules $\mathbf{D}_1, \mathbf{D}_2$ satisfying the following conditions (1), (2B) and (3B):

- (1) $k_i(\mathbf{D}_1) = k_{r-1-i}(\mathbf{D}_2)$ for all $0 \leq i \leq r-1$, and $k_{\frac{r-1}{2}}(\mathbf{D}_1) = k_{\frac{r-1}{2}}(\mathbf{D}_2)$ is even
- (2B) For both $\mathbf{D} = \mathbf{D}_1$ and $\mathbf{D} = \mathbf{D}_2$ we have $\xi(\mathbf{D}) = \prod_{i=0}^{r-1} (k_i(\mathbf{D})!)^{-1}$ and $k_i(\mathbf{D}) = k_{r-1-i}(\mathbf{D})$ for all $1 \leq i \leq \frac{r-1}{2}$
- (3B) $s(\mathbf{D}_2) - s(\mathbf{D}_1) \equiv k_0(\mathbf{D}_1) - k_{r-1}(\mathbf{D}_1) \pmod{p-1}$
- (4) $k_{\bullet}(\mathbf{D}_1) \notin \{(0, \dots, 0), (p-1, \dots, p-1)\}$

Lemma 2.4. *The conjunction of the conditions (1), (2C) and (3C) (resp. (1), (2B) and (3B)) is symmetric in \mathbf{D}_1 and \mathbf{D}_2 .*

PROOF: That each one of the conditions (1), (2C) and (2B) is symmetric even individually is obvious. Now $n(\mathbf{D}) \equiv 0 \pmod{p-1}$ implies $\sum_{i=0}^{r-1} k_i(\mathbf{D}_1) \equiv 0 \pmod{p-1}$. Therefore $s(\mathbf{D}_2) - s(\mathbf{D}_1) \equiv \sum_{i=0}^{r-1} ik_i(\mathbf{D}_1)$ and $k_i(\mathbf{D}_1) = k_{r-1-i}(\mathbf{D}_2)$ for all i (condition (1)) together imply $s(\mathbf{D}_1) - s(\mathbf{D}_2) \equiv \sum_{i=0}^{r-1} ik_i(\mathbf{D}_2)$. Thus condition (3C) is symmetric, assuming condition (1). Similarly, condition (3B) is symmetric, assuming condition (1). \square

Definition: (i) Let $\tilde{\mathfrak{S}}_C(r)$ denote the set of triples (n, s, ξ) with integers $1 \leq n \leq p^r - 2$ and $0 \leq s \leq p - 2$ and scalars $\xi \in k^\times$ such that $n \equiv 0$ modulo $(p - 1)$. Let $\mathfrak{S}_C(r)$ denote the quotient of $\tilde{\mathfrak{S}}_C(r)$ by the involution

$$\left(\sum_{i=0}^{r-1} k_i p^i, s, \xi\right) \mapsto \left(\sum_{i=0}^{r-1} k_{r-i-1} p^i, s + \sum_{i=0}^{r-1} i k_i, \xi\right).$$

(Here and in the following, in the second component we mean the representative modulo $p - 1$ belonging to $[0, p - 2]$.)

(ii) Let r be odd and let $\tilde{\mathfrak{S}}_B(r)$ denote the set of pairs (n, s) with integers $1 \leq n = \sum_{i=0}^{r-1} k_i p^i \leq p^r - 2$ and $0 \leq s \leq p - 2$ such that $n \equiv 0$ modulo $(p - 1)$ and such that $k_i = k_{r-1-i}$ for all $1 \leq i \leq \frac{r-1}{2}$. Let $\mathfrak{S}_B(r)$ denote the quotient of $\tilde{\mathfrak{S}}_B(r)$ by the involution

$$\left(\sum_{i=0}^{r-1} k_i p^i, s\right) \mapsto \left(\sum_{i=0}^{r-1} k_{r-i-1} p^i, s + k_0 - k_{r-1}\right).$$

Lemma 2.5. (i) Sending $\mathbf{D} = \mathbf{D}_1 \oplus \mathbf{D}_2$ to $(n(\mathbf{D}_1), s(\mathbf{D}_1), \xi(\mathbf{D}_1))$ induces a bijection between the set of isomorphism classes of C -symmetric étale (φ^r, Γ) -modules and $\mathfrak{S}_C(r)$.

(ii) Sending $\mathbf{D} = \mathbf{D}_1 \oplus \mathbf{D}_2$ to $(n(\mathbf{D}_1), s(\mathbf{D}_1))$ induces a bijection between the set of isomorphism classes of B -symmetric étale (φ^r, Γ) -modules and $\mathfrak{S}_B(r)$.

PROOF: This follows from Lemma 2.3. \square

Definition: Let r be even. Let $\tilde{\mathfrak{S}}_D(r)$ denote the set of triples (n, s, ξ) with integers $1 \leq n = \sum_{i=0}^{r-1} k_i p^i \leq p^r - 2$ and $0 \leq s \leq p - 2$ and scalars $\xi \in k^\times$ such that $n \equiv 0$ modulo $(p - 1)$ and such that $k_i = k_{i+\frac{r}{2}}$ for all $1 \leq i \leq \frac{r}{2} - 2$. We consider the following permutations ι_0 and ι_1 of $\tilde{\mathfrak{S}}_D(r)$. The value of ι_0 at $(\sum_{i=0}^{r-1} k_i p^i, s, \xi)$ is

$$\left(k_{\frac{r}{2}} + \sum_{i=1}^{\frac{r}{2}-2} k_i p^i + k_{r-1} p^{\frac{r}{2}-1} + k_0 p^{\frac{r}{2}} + \sum_{i=\frac{r}{2}+1}^{r-2} k_i p^i + k_{\frac{r}{2}-1} p^{r-1}, s + \sum_{i=0}^{\frac{r}{2}-1} k_i, \xi\right).$$

The value of ι_1 at $(\sum_{i=0}^{r-1} k_i p^i, s, \xi)$ is

$$\left(\sum_{i=0}^{r-1} k_{r-i-1} p^i, s + \frac{r-2}{4}(k_{\frac{r}{2}} + k_0) + \sum_{i=2}^{\frac{r}{2}-1} (i-1) k_{\frac{r}{2}-i}, \xi\right)$$

if $\frac{r}{2}$ is odd, whereas if $\frac{r}{2}$ is even the value is

$$\left(k_{\frac{r}{2}-1} + \sum_{i=1}^{\frac{r}{2}-2} k_{r-i-1} p^i + k_{\frac{r}{2}} p^{\frac{r}{2}-1} + k_{r-1} p^{\frac{r}{2}} + \sum_{i=\frac{r}{2}+1}^{r-2} k_{r-i-1} p^i + k_0 p^{r-1}, s + \left(\frac{r}{4}-1\right) k_{\frac{r}{2}} + \frac{r}{4} k_0 + \sum_{i=2}^{\frac{r}{2}-1} (i-1) k_{\frac{r}{2}-i} p^i, \xi\right).$$

It is straightforward to check that $\iota_0^2 = \text{id}$ and $\iota_0\iota_1 = \iota_1\iota_0$, and moreover that $\iota_1^2 = \text{id}$ if $\frac{r}{2}$ is odd, but $\iota_1^2 = \iota_0$ if $\frac{r}{2}$ is even. In either case, the subgroup $\langle \iota_0, \iota_1 \rangle$ of $\text{Aut}(\tilde{\mathfrak{S}}_D(r))$ generated by ι_0 and ι_1 is commutative and contains 4 elements. We let $\mathfrak{S}_D(r)$ denote the quotient of $\tilde{\mathfrak{S}}_D(r)$ by the action of $\langle \iota_0, \iota_1 \rangle$.

Definition: Let r be even. We say that an étale (φ^r, Γ) -module \mathbf{D} over $k_{\mathcal{E}}$ is D -symmetric if it admits a direct sum decomposition $\mathbf{D} = \mathbf{D}_{11} \oplus \mathbf{D}_{12} \oplus \mathbf{D}_{21} \oplus \mathbf{D}_{22}$ with one-dimensional étale (φ^r, Γ) -modules $\mathbf{D}_{11}, \mathbf{D}_{12}, \mathbf{D}_{21}, \mathbf{D}_{22}$ satisfying the following conditions:

- (1) For all $1 \leq i \leq \frac{r}{2} - 2$ and all $1 \leq s, t \leq 2$ we have $k_i(\mathbf{D}_{st}) = k_{\frac{r}{2}+i}(\mathbf{D}_{st})$
- (2) For all $1 \leq i \leq \frac{r}{2} - 2$ we have $k_i(\mathbf{D}_{11}) = k_i(\mathbf{D}_{12})$ and $k_i(\mathbf{D}_{21}) = k_i(\mathbf{D}_{22})$
- (3) For $j = 1$ and $j = 2$ we have

$$k_0(\mathbf{D}_{j1}) = k_{\frac{r}{2}}(\mathbf{D}_{j2}), \quad k_{\frac{r}{2}}(\mathbf{D}_{j1}) = k_0(\mathbf{D}_{j2}), \quad k_{\frac{r}{2}-1}(\mathbf{D}_{j1}) = k_{r-1}(\mathbf{D}_{j2}), \quad k_{r-1}(\mathbf{D}_{j1}) = k_{\frac{r}{2}-1}(\mathbf{D}_{j2}).$$

(4)

$$k_i(\mathbf{D}_{11}) = k_{r-i-1}(\mathbf{D}_{21}) \quad \text{and} \quad k_i(\mathbf{D}_{12}) = k_{r-i-1}(\mathbf{D}_{22})$$

if $i \in [0, r-1]$ and $\frac{r}{2}$ is odd, or if $i \in [1, \frac{r}{2} - 2] \cup [\frac{r}{2} + 1, r-2]$ and $\frac{r}{2}$ is even. Moreover, if $\frac{r}{2}$ is even then

$$k_0(\mathbf{D}_{11}) = k_{r-1}(\mathbf{D}_{21}), \quad k_{\frac{r}{2}-1}(\mathbf{D}_{11}) = k_0(\mathbf{D}_{21}), \quad k_{\frac{r}{2}}(\mathbf{D}_{11}) = k_{\frac{r}{2}-1}(\mathbf{D}_{21}), \quad k_{r-1}(\mathbf{D}_{11}) = k_{\frac{r}{2}}(\mathbf{D}_{21})$$

$$(5) \quad \xi(\mathbf{D}_{11}) = \xi(\mathbf{D}_{12}) = \xi(\mathbf{D}_{21}) = \xi(\mathbf{D}_{22})$$

(6) Modulo $(p-1)$ we have

$$s(\mathbf{D}_{j2}) - s(\mathbf{D}_{j1}) \equiv \sum_{i=0}^{\frac{r}{2}-1} k_i(\mathbf{D}_{j1}) \quad \text{for } j = 1, 2$$

$$s(\mathbf{D}_{21}) - s(\mathbf{D}_{11}) \equiv \begin{cases} \frac{r-2}{4}(k_{\frac{r}{2}}(\mathbf{D}_{11}) + k_0(\mathbf{D}_{11})) + \sum_{i=2}^{\frac{r}{2}-1} (i-1)k_{\frac{r}{2}-i}(\mathbf{D}_{11}) & : \frac{r}{2} \text{ is odd} \\ (\frac{r}{4}-1)k_{\frac{r}{2}}(\mathbf{D}_{11}) + \frac{r}{4}k_0(\mathbf{D}_{11}) + \sum_{i=2}^{\frac{r}{2}-1} (i-1)k_{\frac{r}{2}-i}(\mathbf{D}_{11}) & : \frac{r}{2} \text{ is even} \end{cases}$$

$$(7) \quad k_{\bullet}(\mathbf{D}_{11}) \notin \{(0, \dots, 0), (p-1, \dots, p-1)\}$$

Lemma 2.6. Sending $\mathbf{D} = \mathbf{D}_{11} \oplus \mathbf{D}_{12} \oplus \mathbf{D}_{21} \oplus \mathbf{D}_{22}$ to $(n(\mathbf{D}_{11}), s(\mathbf{D}_{11}), \xi(\mathbf{D}_{11}))$ induces a bijection between the set of isomorphism classes of D -symmetric étale (φ^r, Γ) -modules and $\mathfrak{S}_D(r)$.

PROOF: Again we use Lemma 2.3. For a one-dimensional étale (φ^r, Γ) -module \mathbf{D} over $k_{\mathcal{E}}$ put $\alpha(\mathbf{D}) = (n(\mathbf{D}), s(\mathbf{D}), \xi(\mathbf{D}))$. If $\mathbf{D} = \mathbf{D}_{11} \oplus \mathbf{D}_{12} \oplus \mathbf{D}_{21} \oplus \mathbf{D}_{22}$ is D -symmetric as above, then $\alpha(\mathbf{D}_{st})$ is an element of $\tilde{\mathfrak{S}}_D(r)$, for all $1 \leq s, t \leq 2$. Moreover, it is straightforward to check $\iota_0(\alpha(\mathbf{D}_{11})) = \alpha(\mathbf{D}_{12})$, $\iota_0(\alpha(\mathbf{D}_{21})) = \alpha(\mathbf{D}_{22})$, $\iota_1(\alpha(\mathbf{D}_{11})) = \alpha(\mathbf{D}_{21})$ and

$\iota_1(\alpha(\mathbf{D}_{12})) = \alpha(\mathbf{D}_{22})$. It follows that the above map is well defined and bijective. \square

Definition: We say that an étale (φ^r, Γ) -module \mathbf{D} over $k_{\mathcal{E}}$ is A -symmetric if \mathbf{D} admits a direct sum decomposition $\mathbf{D} = \bigoplus_{i=0}^{r-1} \mathbf{D}_i$ with one-dimensional étale (φ^r, Γ) -modules \mathbf{D}_i satisfying

$$k_i(\mathbf{D}_j) = k_{i-j}(\mathbf{D}_0), \quad \xi(\mathbf{D}_j) = \xi(\mathbf{D}_0), \quad s(\mathbf{D}_0) - s(\mathbf{D}_j) \equiv \sum_{i=1}^j k_{-i}(\mathbf{D}_0) \text{ modulo } (p-1)$$

for all i, j (where we understand the sub index in $k_?$ as the unique representative in $[0, r-1]$ modulo r), and moreover

$$k_{\bullet}(\mathbf{D}_0) \notin \{(0, \dots, 0), (p-1, \dots, p-1)\}.$$

3 Semiinfinite chamber galleries and the functor \mathbf{D}

3.1 Power multiplicative elements in the extended affine Weyl group

Let G be the group of \mathbb{Q}_p -rational points of a \mathbb{Q}_p -split connected reductive group over \mathbb{Q}_p . Fix a maximal \mathbb{Q}_p -split torus T in G , let $N(T)$ be its normalizer in G . Let Φ denote the set of roots of T . For $\alpha \in \Phi$ let N_{α} be the corresponding root subgroup in G . Choose a positive system Φ^+ in Φ , let $\Delta \subset \Phi^+$ be the set of simple roots. Let $N = \prod_{\alpha \in \Phi^+} N_{\alpha}$.

Let X denote the semi simple Bruhat-Tits building of G , let A denote its apartment corresponding to T . Our notational and terminological convention is that the facets of A or X are *closed* in X (i.e. *contain* all their faces (the lower dimensional facets at their boundary)). A chamber is a facet of codimension 0. For a chamber D in A let I_D be the Iwahori subgroup in G fixing D . We notice that $I_D \cap N = \prod_{\alpha \in \Phi^+} I_D \cap N_{\alpha}$.

Fix a special vertex x_0 in A , let K be the corresponding hyperspecial maximal compact open subgroup in G . Let $T_0 = T \cap K$ and $N_0 = N \cap K$. We have the isomorphism $T/T_0 \cong X_*(T)$ sending $\xi \in X_*(T)$ to the class of $\xi(p) \in T$. Let $I \subset K$ be the Iwahori subgroup determined by Φ^+ . [If $\text{red} : K \rightarrow \overline{K}$ denotes the reduction map onto the reductive (over \mathbb{F}_p) quotient \overline{K} of K , then $I = \text{red}^{-1}(\text{red}(T_0 N_0))$.] Let $C \subset A$ be the chamber fixed by I . Thus $I = I_C$ and $I \cap N = N_0 = \prod_{\alpha \in \Phi^+} N_0 \cap N_{\alpha}$.

We are interested in semiinfinite chamber galleries

$$(3) \quad C^{(0)}, C^{(1)}, C^{(2)}, C^{(3)}, \dots$$

in A such that $C = C^{(0)}$ (and thus $I = I_{C^{(0)}}$) and such that, setting

$$N_0^{(i)} = I_{C^{(i)}} \cap N,$$

we have $N_0 = N_0^{(0)}$ and

$$(4) \quad N_0^{(0)} \supset N_0^{(1)} \supset N_0^{(2)} \supset N_0^{(3)} \supset \dots \quad \text{with } [N_0^{(i)} : N_0^{(i+1)}] = p \text{ for all } i \geq 0.$$

For any two chambers $D_1 \neq D_2$ in A sharing a common facet of codimension 1 there is a uniquely determined $\delta \in \Phi$ with $I_{D_1} \cap N_{\delta'} = I_{D_2} \cap N_{\delta'}$ for $\delta' \notin \{\delta, -\delta\}$, with $I_{D_1} \cap N_{-\delta} \subset I_{D_2} \cap N_{-\delta}$ and with $I_{D_2} \cap N_{\delta} \subset I_{D_1} \cap N_{\delta}$, and both these inclusions are of index p . Moreover, the one-codimensional facet shared by D_1 and D_2 is contained in a translate of the hyperplane corresponding to δ (or equivalently, corresponding to $-\delta$).^{*}

Applying this remark to $D_1 = C^{(i)}$, $D_2 = C^{(i+1)}$ and putting $\alpha^{(i)} = \delta$ we see that condition (4) precisely means that $\alpha^{(i)} \in \Phi^+$ for all $i \geq 0$. Conversely, if condition (4) holds true then the sequence $\alpha^{(0)}, \alpha^{(1)}, \alpha^{(2)}, \alpha^{(3)}, \dots$ in Φ^+ can be characterized as follows: Setting

$$e[i, \alpha] = |\{0 \leq j \leq i-1 \mid \alpha = \alpha^{(j)}\}|$$

for $i \geq 0$ and $\alpha \in \Phi^+$, we have

$$N_0^{(i)} = \prod_{\alpha \in \Phi^+} (N_0 \cap N_{\alpha})^{p^{e[i, \alpha]}}.$$

Suppose that the center Z of G is connected. Then G/Z is a semisimple group of adjoint type with maximal torus $\check{T} = T/Z$. Let $\check{T}_0 = T_0/(T_0 \cap Z) \subset \check{T}$. The extended affine Weyl group $\widehat{W} = N(\check{T})/\check{T}_0$ can be identified with the semidirect product between the finite Weyl group $W = N(\check{T})/\check{T} = N(T)/T$ and $X_*(\check{T})$. We identify $A = X_*(\check{T}) \otimes \mathbb{R}$ such that $x_0 \in A$ corresponds to the origin in the \mathbb{R} -vector space $X_*(\check{T}) \otimes \mathbb{R}$. We then regard \widehat{W} as acting on A through affine transformations. We regard $\Delta \subset X^*(T)$ as a subset of $X^*(\check{T})$. We usually enumerate the elements of Δ as $\alpha_1, \dots, \alpha_d$, and we enumerate the corresponding simple reflection $s_{\alpha} \in W$ for $\alpha \in \Delta$ as s_1, \dots, s_d with $s_i = s_{\alpha_i}$. We assume that the root system Φ is irreducible. Let $\alpha_0 \in \Phi$ be the *negative* of the highest root. Let s_{α_0} be the corresponding reflection in the finite Weyl group W ; define the affine reflection $s_0 = t_{\alpha_0^\vee} \circ s_{\alpha_0} \in \widehat{W}$, where $t_{\alpha_0^\vee}$ denotes the translation by the coroot $\alpha_0^\vee \in A$ of α_0 . The affine Weyl group W_{aff} is the subgroup of \widehat{W} generated by s_0, s_1, \dots, s_d ; in fact it is a Coxeter group with these Coxeter generators. The corresponding length function ℓ on W_{aff} extends to \widehat{W} .

The above discussion shows that if the gallery (3) satisfies condition (4) and if $w \in \widehat{W}$ satisfies $C^{(i)} = wC$ for some $i \geq 0$, then

$$p^{\ell(w)} = p^i = p^{[N_0 : I_{C^{(i)}} \cap N_0]}.$$

^{*}Pick $g \in N(T)$ acting as the reflection in the affine hyperplane which contains the one-codimensional facet shared by D_1 and D_2 . Then $gD_1 = D_2$ and hence $I_{D_2} = gI_{D_1}g^{-1}$, so we can read off all these claims.

Let $X_*(\check{T})_+$ denote the set of dominant cocharacters. [Let $T_+ = \{t \in T \mid tN_0t^{-1} \subset N_0\}$, then $X_*(\check{T})_+$ is the image of T_+ under the map $T_+ \subset T \rightarrow T/T_0 \cong X_*(T) \rightarrow X_*(\check{T})$.] The monoid $X_*(\check{T})_+$ is free and has a distinguished basis ∇ , the set of fundamental coweights. The cone \mathcal{C} (vector chamber) in A with origin in x_0 which is spanned by all the $-\xi$ for $\xi \in \nabla$ contains C , and C is precisely the 'top' chamber of this cone. The reflections s_0, s_1, \dots, s_d are precisely the reflections in the affine hyperplanes (walls) of A which contain a codimension-1-face of C .

Definition: We say that $w \in \widehat{W}$ is power multiplicative if $\ell(w^m) = m \cdot \ell(w)$ for all $m \geq 0$.[†]

Of course, any element in the image of $T \rightarrow N(\check{T}) \rightarrow \widehat{W} = N(\check{T})/\check{T}_0$ is power multiplicative because it acts on A by translation (and $\ell(w)$ is the gallery distance between C and wC).

Suppose we are given a fundamental coweight $\tau \in \nabla$ and some non trivial element $\phi \in \widehat{W}$ satisfying the following conditions:

- (a) ϕ is power multiplicative,
- (b) τ is minuscule, i.e. we have $\langle \alpha, \tau \rangle \in \{0, 1\}$ for all $\alpha \in \Phi^+$,
- (c) viewing τ via the embedding $X_*(\check{T}) \subset \widehat{W}$ as an element of \widehat{W} , we have

$$(5) \quad \phi^{\mathbb{N}} \cap \tau^{\mathbb{N}} \neq \emptyset.$$

Lemma 3.1. *Let ϕ and τ be as above. Write $\phi = \phi'v$ with $\phi' \in W_{\text{aff}}$ and $v \in \widehat{W}$ with $vC = C$. Choose a reduced expression*

$$\phi' = s_{\beta(1)} \cdots s_{\beta(r)}$$

of ϕ' with some function $\beta : \{1, \dots, r\} \rightarrow \{0, \dots, d\}$ (with $r = \ell(\phi) = \ell(\phi')$) and put

$$(6) \quad C^{(ar+b)} = \phi^a s_{\beta(1)} \cdots s_{\beta(b)} C$$

for $a, b \in \mathbb{Z}_{\geq 0}$ with $0 \leq b < r$. Lift $\tau \in \nabla \subset X_(\check{T})$ to some element of $X_*(T)$ and denote again by τ the corresponding homomorphism $\mathbb{Z}_p^\times \rightarrow T_0$. Then we have:*

(i) *The sequence*

$$C = C^{(0)}, C^{(1)}, C^{(2)}, \dots$$

satisfies hypothesis (4). In particular we may define $\alpha^{(j)} \in \Phi^+$ for all $j \geq 0$.

(ii) *For any $j \geq 0$ we have $\alpha^{(j)} \circ \tau = \text{id}_{\mathbb{Z}_p^\times}$.*

(iii) *For any lifting $\phi \in N(T)$ of $\phi \in \widehat{W}$ we have $\tau(a)\phi = \phi\tau(a)$ in $N(T)$, for all $a \in \mathbb{Z}_p^\times$.*

[†]After finishing this article we found out that power multiplicative elements of \widehat{W} have been studied e.g. in [3] under the name of *straight* elements.

PROOF: Given a chamber gallery (3) in A with $C = C^{(0)}$ and $N_0 = N_0^{(0)}$, condition (4) is equivalent with saying that (3) is a minimal chamber gallery in the cone \mathcal{C} , i.e. all $C^{(i)}$ are contained in \mathcal{C} , and the gallery distance from $C = C^{(0)}$ to $C^{(i)}$ is i . Conversely, notice that if $C^{(i)}$ belongs to \mathcal{C} and the gallery distance from $C = C^{(0)}$ to $C^{(i)}$ is i , then also all $C^{(j)}$ for $0 \leq j \leq i$ belong to \mathcal{C} . (Otherwise the gallery would have to cross one of the bounding hyperplanes of \mathcal{C} and then cross it again later in the reverse direction; but a minimal gallery never crosses a wall (hyperplane translate) back and forth.) Recall that, assuming condition (4), the set $\{\alpha^{(j)} \mid j \geq 0\}$ is precisely the set of all $\alpha \in \Phi^+$ such that the gallery (3) crosses a translate of the hyperplane corresponding to α . If there is some $\epsilon \in X_*(T)$ and some $s \in \mathbb{N}$ such that $C^{(i+s)} = \epsilon C^{(i)}$ for all i , then, since the action of ϵ is by translation on A , the latter set is the same as the set of all $\alpha \in \Phi^+$ such that the gallery $C = C^{(0)}, C^{(1)}, \dots, C^{(s)}$ crosses a translate of the hyperplane corresponding to α .

Now let (3) be the chamber gallery defined by formula (6). (i) If $n \in \mathbb{N}$ is such that $\phi^n \in X_*(T)$, then the gallery distance between C and $\phi^n C$ is $\ell(\phi^n)$. By construction, the gallery $C = C^{(0)}, C^{(1)}, \dots, C^{(n\ell(\phi))} = \phi^n C$ has length $n\ell(\phi)$, i.e. length $\ell(\phi^n)$ as ϕ is power multiplicative. Hence it must be a minimal chamber gallery. As τ is minuscule, it is in particular a dominant coweight. Therefore it follows from hypothesis (5) that also some power ϕ^n of ϕ is a dominant coweight. We thus obtain that $C^{(n'\ell(\phi))} = \phi^{n'} C$ lies in \mathcal{C} for each $n' \in \mathbb{N}$ divisible by n , but then, as the gallery is minimal and \mathcal{C} is a cone, $C^{(i)}$ lies in \mathcal{C} for each $i \geq 0$. We get statement (i) in view of the preceeding discussion.

(ii) The preceeding discussion shows $\{\alpha^{(j)} \mid j \geq 0\} = \{\alpha \in \Phi^+ \mid \langle \alpha, \phi^m \rangle \neq 0\}$ for any $m \in \mathbb{N}$ such that ϕ^m belongs to $X_*(T)$. As ϕ is power multiplicative, hypothesis (5) furthermore gives

$$\{\alpha^{(j)} \mid j \geq 0\} = \{\alpha \in \Phi^+ \mid \langle \alpha, \phi^m \rangle \neq 0\} = \{\alpha \in \Phi^+ \mid \langle \alpha, \tau \rangle \neq 0\}$$

and as τ is minuscule this is the set

$$\{\alpha \in \Phi^+ \mid \langle \alpha, \tau \rangle = 1\} = \{\alpha \in \Phi^+ \mid \alpha \circ \tau = \text{id}_{\mathbb{Z}_p^\times}\}.$$

(iii) By hypothesis (5) we have $\tau^m = \phi^n$ for some $m, n \in \mathbb{N}$. We deduce $\tau^m = \phi \tau^m \phi^{-1} = (\phi \tau \phi^{-1})^m$ and hence also $\tau = \phi \tau \phi^{-1}$ as τ and $\phi \tau \phi^{-1}$ belong to the free abelian group $X_*(T)$. Thus $\tau \phi = \phi \tau$ in \widehat{W} which implies claim (iii). \square

Remark: For a given minuscule fundamental coweight $\tau \in \nabla$ any positive power τ^m of τ belongs to $X_*(\check{T})$, and so $\phi = \tau^m$ satisfies the assumptions of Lemma 3.1. However, for our purposes it is of interest to find ϕ (as in Lemma 3.1, possibly also required to project to W_{aff}) of small length; the minimal positive power of τ belonging to $X_*(\check{T})$ is usually not optimal in this sense.

3.2 The functor D

By I_0 we denote the pro- p -Iwahori subgroup contained in I . We often read $\overline{T} = T_0/T_0 \cap I_0$ as a subgroup of T_0 by means of the Teichmüller character. Conversely, we read characters of \overline{T} also as characters of T_0 (and do not introduce another name for these inflations).

Let $\text{ind}_{I_0}^G \mathbf{1}_{\mathfrak{o}}$ denote the \mathfrak{o} -module of \mathfrak{o} -valued compactly supported functions f on G such that $f(ig) = f(g)$ for all $g \in G$, all $i \in I_0$. It is a G -representation by means of $(g'f)(g) = f(gg')$ for $g, g' \in G$. Let

$$\mathcal{H}(G, I_0) = \text{End}_{\mathfrak{o}[G]}(\text{ind}_{I_0}^G \mathbf{1}_{\mathfrak{o}})^{\text{op}}$$

denote the corresponding pro- p -Iwahori Hecke algebra with coefficients in \mathfrak{o} . For a subset H of G let χ_H denote the characteristic function of H . For $g \in G$ let $T_g \in \mathcal{H}(G, I_0)$ denote the Hecke operator corresponding to the double coset $I_0 g I_0$. It sends $f : G \rightarrow \mathfrak{o}$ to

$$T_g(f) : G \longrightarrow \mathfrak{o}, \quad h \mapsto \sum_{x \in I_0 \backslash G} \chi_{I_0 g I_0}(hx^{-1})f(x).$$

Let $\text{Mod}^{\text{fin}}(\mathcal{H}(G, I_0))$ denote the category of $\mathcal{H}(G, I_0)$ -modules which as \mathfrak{o} -modules are of finite length. We write $\mathcal{H}(G, I_0)_k = \mathcal{H}(G, I_0) \otimes_{\mathfrak{o}} k$. Given liftings $\dot{s} \in N(T)$ of all $s \in S$ we let $\mathcal{H}(G, I_0)_{\text{aff}, k}$ denote the k -subalgebra of $\mathcal{H}(G, I_0)_k$ generated by the $T_{\dot{s}}$ for all $s \in S$ and the T_t for $t \in \overline{T}$.

Suppose we are given a reduced expression

$$(7) \quad \phi = \epsilon \dot{s}_{\beta(1)} \cdots \dot{s}_{\beta(r)}$$

(some function $\beta : \{1, \dots, r = \ell(\phi)\} \rightarrow \{0, \dots, d\}$, some $\epsilon \in Z$) of a power multiplicative element $\phi \in N(T)$, some power of which maps to a dominant coweight in $N(T)/ZT_0$. Put

$$C^{(ar+b)} = \phi^a s_{\beta(1)} \cdots s_{\beta(b)} C$$

for $a, b \in \mathbb{Z}_{\geq 0}$ with $0 \leq b < r$. Then, by power multiplicativity of ϕ , the sequence (3) thus defined satisfies property (4), cf. Lemma 3.1. Therefore we may use it to place ourselves into the setting (and notations) of [2], as follows.

We define the half tree Y whose edges are the elements in the N_0 -orbits of the $C^{(i)} \cap C^{(i+1)}$ and whose vertices are the elements in the N_0 -orbits of the $C^{(i)}$. We choose an isomorphism $\Theta : Y \cong \mathfrak{X}_+$ with the $[\mathfrak{N}_0, \varphi, \Gamma]$ -equivariant half sub tree \mathfrak{X}_+ of the Bruhat Tits tree of $\text{GL}_2(\mathbb{Q}_p)$, satisfying the requirements of Theorem 3.2 of loc.cit.. It sends the edge $C^{(i)} \cap C^{(i+1)}$ (resp. the vertex $C^{(i)}$) of Y to the edge \mathfrak{e}_{i+1} (resp. the vertex \mathfrak{v}_i) of \mathfrak{X}_+ . The half tree $\overline{\mathfrak{X}}_+$ is obtained from \mathfrak{X}_+ by removing the 'loose' edge \mathfrak{e}_0 .

To an $\mathcal{H}(G, I_0)$ -module M we associate the G -equivariant (partial) coefficient system \mathcal{V}_M^X on X . Briefly, its value at the chamber C is $\mathcal{V}_M^X(C) = M$. The transition maps

$\mathcal{V}_M^X(D) \rightarrow \mathcal{V}_M^X(F)$ for chambers (codimension-0-facets) D and codimension-1-facets F with $F \subset D$ are injective, and $\mathcal{V}_M^X(F)$ for any such F is the sum of the images of the $\mathcal{V}_M^X(D) \rightarrow \mathcal{V}_M^X(F)$ for all D with $F \subset D$.

The pushforward $\Theta_* \mathcal{V}_M$ of the restriction of \mathcal{V}_M^X to Y is a $[\mathfrak{N}_0, \varphi^r, \Gamma_0]$ -equivariant coefficient system on \mathfrak{X}_+ (and by further restriction on $\overline{\mathfrak{X}}_+$). This leads to the exact functor

$$(8) \quad M \mapsto \mathbf{D}(\Theta_* \mathcal{V}_M)$$

$$\mathbf{D}(\Theta_* \mathcal{V}_M) = H_0(\overline{\mathfrak{X}}_+, \Theta_* \mathcal{V}_M) \otimes_{\mathcal{O}_{\mathcal{E}}^+} \mathcal{O}_{\mathcal{E}} = H_0(\mathfrak{X}_+, \Theta_* \mathcal{V}_M) \otimes_{\mathcal{O}_{\mathcal{E}}^+} \mathcal{O}_{\mathcal{E}}$$

from $\text{Mod}^{\text{fin}}(\mathcal{H}(G, I_0))$ to the category of (φ^r, Γ_0) -modules over $\mathcal{O}_{\mathcal{E}}$, where $r = \ell(\phi)$. If in addition we are given a homomorphism $\tau : \mathbb{Z}_p^\times \rightarrow T_0$ satisfying the conclusions of Lemma 3.1 (with respect to ϕ), then this functor in fact takes values in the category of (φ^r, Γ) -modules over $\mathcal{O}_{\mathcal{E}}$.

For $0 \leq i \leq r-1$ we put

$$y_i = \dot{s}_{\beta(1)} \cdots \dot{s}_{\beta(i+1)} \dot{s}_{\beta(i)}^{-1} \cdots \dot{s}_{\beta(1)}^{-1}.$$

Lemma 3.2. (a) $y_i = y_{i-1} \cdots y_0 \dot{s}_{\beta(i+1)} y_0^{-1} \cdots y_{i-1}^{-1}$ and $\phi = \epsilon y_{r-1} \cdots y_0$.

(b) y_i is the affine reflection in the wall passing through $C^{(i)} \cap C^{(i+1)}$.

PROOF: To see (b) observe that y_i indeed is a reflection, and that it sends $C^{(i)}$ to $C^{(i+1)}$. \square

3.3 Supersingular modules

For $\alpha \in \Phi$ we denote by α^\vee the associated coroot. For any $\alpha \in \Phi$ there is a corresponding homomorphism of algebraic groups $\iota_\alpha : \text{SL}_2(\mathbb{Q}_p) \rightarrow G$ as described in [5], Ch.II, section 1.3. The element $\iota_\alpha(\nu)$ belongs to $I \cap N_\alpha$ and generates it as a topological group. For $x \in \mathbb{F}_p^\times \subset \mathbb{Z}_p^\times$ (via the Teichmüller character) we have $\alpha^\vee(x) = \iota_\alpha(h(x)) \in T$.

For $0 \leq i \leq d$ let $\overline{T}_{\alpha_i^\vee}$ denote the subgroup of \overline{T} given by the \mathbb{F}_p -valued points of the schematic closure of $\alpha_i^\vee(\overline{\mathbb{F}}_p^\times)$ (in the obvious \mathbb{F}_p -group scheme (split torus) underlying \overline{T}). For a character $\lambda : \overline{T} \rightarrow k^\times$ let S_λ be the subset of $S = \{s_0, \dots, s_d\}$ consisting of all s_i such that $\lambda|_{\overline{T}_{\alpha_i^\vee}}$ is trivial. Given λ and a subset \mathcal{J} of S_λ there is a uniquely determined character

$$\chi_{\lambda, \mathcal{J}} : \mathcal{H}(G, I_0)_{\text{aff}, k} \longrightarrow k$$

which sends T_t to $\lambda(t^{-1})$ for $t \in \overline{T}$, which sends T_s to 0 for $s \in S - \mathcal{J}$ and which sends T_s to -1 for $s \in \mathcal{J}$ (see [8] Proposition 2). For $0 \leq i \leq d$ we define a number

$0 \leq k_i = k_i(\lambda, \mathcal{J}) \leq p-1$ such that

$$(9) \quad \lambda(\alpha_i^\vee(x)) = x^{k_i} \quad \text{for all } x \in \mathbb{F}_p^\times,$$

as follows. If $s_i \in \mathcal{J}$ put $k_i = p-1$. Otherwise let k_i be the unique integer in $[0, p-2]$ satisfying formula (9).

Vignéras defined the notion of a *supersingular* $\mathcal{H}(G, I_0)_k$ -module, see [8], [7]. Let $\mathcal{H}(G, I_0)'_{\text{aff}, k}$ denote the k -subalgebra of $\mathcal{H}(G, I_0)_k$ generated by $\mathcal{H}(G, I_0)_{\text{aff}, k}$ together with all T_z for $z \in Z$.

Theorem 3.3. (*Vignéras, Ollivier*)

(a) Let $\chi : \mathcal{H}(G, I_0)'_{\text{aff}, k} \rightarrow k$ be a character extending $\chi_{\lambda, \mathcal{J}} : \mathcal{H}(G, I_0)_{\text{aff}, k} \rightarrow k$ for some λ, \mathcal{J} . If $k_\bullet \notin \{(0, \dots, 0), (p-1, \dots, p-1)\}$ then the $\mathcal{H}(G, I_0)_k$ -module $\mathcal{H}(G, I_0)_k \otimes_{\mathcal{H}(G, I_0)'_{\text{aff}, k}} \chi$ is supersingular.

(b) Any supersingular $\mathcal{H}(G, I_0)_k$ -module of finite length contains a character $\chi_{\lambda, \mathcal{J}} : \mathcal{H}(G, I_0)_{\text{aff}, k} \rightarrow k$ for some λ, \mathcal{J} with $k_\bullet \notin \{(0, \dots, 0), (p-1, \dots, p-1)\}$.

PROOF: See [7] Theorem 5.14, Corollary 5.16, at least for the case where the root system underlying G is irreducible. See [9] for more general statements. \square

It follows that the simple supersingular $\mathcal{H}(G, I_0)_k$ -modules are precisely the simple quotients of the $\mathcal{H}(G, I_0)_k$ -modules $\mathcal{H}(G, I_0)_k \otimes_{\mathcal{H}(G, I_0)'_{\text{aff}, k}} \chi$ appearing in statement (a) of Theorem 3.3. We call the latter ones *standard supersingular* $\mathcal{H}(G, I_0)_k$ -modules. In fact, it is easy to see that these are simple themselves if the pair (k_\bullet, \mathcal{J}) satisfies a certain non-periodicity property.

4 Classical matrix groups

For $m \in \mathbb{N}$ let $E_m \in \text{GL}_m$ denote the identity matrix and let E_d^* denote the standard antidiagonal element in GL_d (i.e. the permutation matrix of maximal length). Let

$$\widehat{S}_m = \begin{pmatrix} & E_m \\ -E_m & \end{pmatrix}, \quad S_m = \begin{pmatrix} & E_m \\ E_m & \end{pmatrix}.$$

For $1 \leq i, j \leq m$ with $i \neq j$ let $\epsilon_{i,j} \in \text{GL}_m$ denote the matrix with entries 1 on the diagonal and at the spot (i, j) and with entries 0 otherwise.

4.1 Affine root system \tilde{C}_d

Assume $d \geq 2$. Here W_{aff} is the Coxeter group with generators s_0, s_1, \dots, s_d and relations

$$(10) \quad (s_0 s_1)^4 = (s_{d-1} s_d)^4 = 1 \quad \text{and} \quad (s_{i-1} s_i)^3 = 1 \quad \text{for } 2 \leq i \leq d-1$$

and moreover $(s_i s_j)^2 = 1$ for all other pairs $i < j$, and $s_i^2 = 1$ for all i . In the extended affine Weyl group \widehat{W} we find (cf. [4]) an element u of length 0 with

$$(11) \quad u^2 = 1 \quad \text{and} \quad u s_i u = s_{d-i} \quad \text{for } 0 \leq i \leq d.$$

(We have $\widehat{W} = W_{\text{aff}} \rtimes W_\Omega$ with the two-element subgroup $W_\Omega = \{1, u\}$.) Consider the symplectic and the general symplectic group

$$\text{Sp}_{2d}(\mathbb{Q}_p) = \{A \in \text{GL}_{2d}(\mathbb{Q}_p) \mid {}^T A \widehat{S}_d A = \widehat{S}_d\},$$

$$G = \text{GSp}_{2d}(\mathbb{Q}_p) = \{A \in \text{GL}_{2d}(\mathbb{Q}_p) \mid {}^T A \widehat{S}_d A = \kappa(A) \widehat{S}_d \text{ for some } \kappa(A) \in \mathbb{Q}_p^\times\}.$$

Let T denote the maximal torus in G consisting of all diagonal matrices in G . For $1 \leq i \leq d$ let

$$e_i : T \cap \text{Sp}_{2d}(\mathbb{Q}_p) \longrightarrow \mathbb{Q}_p^\times, \quad \text{diag}(x_1, \dots, x_{2d}) \mapsto x_i.$$

For $1 \leq i, j \leq d$ and $\epsilon_1, \epsilon_2 \in \{\pm 1\}$ we thus obtain characters (using additive notation as usual) $\epsilon_1 e_i + \epsilon_2 e_j : T \cap \text{Sp}_{2d}(\mathbb{Q}_p) \longrightarrow \mathbb{Q}_p^\times$. We extend these latter ones to T by setting

$$\epsilon_1 e_i + \epsilon_2 e_j : T \longrightarrow \mathbb{Q}_p^\times, \quad A = \text{diag}(x_1, \dots, x_{2d}) \mapsto x_i^{\epsilon_1} x_j^{\epsilon_2} \kappa(A)^{\frac{-\epsilon_1 - \epsilon_2}{2}}.$$

For $i = j$ and $\epsilon = \epsilon_1 = \epsilon_2$ we simply write $\epsilon 2e_i$. Then $\Phi = \{\pm e_i \pm e_j \mid i \neq j\} \cup \{\pm 2e_i\}$ is the root system of G with respect to T . It is of type C_d . We choose the positive system $\Phi^+ = \{e_i \pm e_j \mid i < j\} \cup \{2e_i \mid 1 \leq i \leq d\}$ with corresponding set of simple roots $\Delta = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \dots, \alpha_{d-1} = e_{d-1} - e_d, \alpha_d = 2e_d\}$. The negative of the highest root is $\alpha_0 = -2e_1$. For $0 \leq i \leq d$ we have the following explicit formula for $\alpha_i^\vee = (\alpha_i)^\vee$:

$$(12) \quad \alpha_i^\vee(x) = \begin{cases} \text{diag}(x^{-1}, E_{d-1}, x, E_{d-1}) & : & i = 0 \\ \text{diag}(E_{i-1}, x, x^{-1}, E_{d-i-1}, E_{i-1}, x^{-1}, x, E_{d-i-1}) & : & 1 \leq i \leq d-1 \\ \text{diag}(E_{d-1}, x, E_{d-1}, x^{-1}) & : & i = d \end{cases}$$

For $\alpha \in \Phi$ let N_α^0 be the subgroup of the corresponding root subgroup N_α of G all of whose elements belong to $\text{GL}_{2d}(\mathbb{Z}_p)$. Let I_0 denote the pro- p -Iwahori subgroup generated by the N_α^0 for all $\alpha \in \Phi^+$, by the $(N_\alpha^0)^p$ for all $\alpha \in \Phi^- = \Phi - \Phi^+$, and by the maximal pro- p -subgroup of T_0 . Let I denote the Iwahori subgroup of G containing I_0 . Let N_0 be the subgroup of G generated by all N_α^0 for $\alpha \in \Phi^+$.

For $1 \leq i \leq d-1$ define the block diagonal matrix

$$\dot{s}_i = \text{diag}(E_{i-1}, \widehat{S}_1, E_{d-i-1}, E_{i-1}, \widehat{S}_1, E_{d-i-1})$$

and furthermore

$$\dot{s}_d = \begin{pmatrix} E_{d-1} & & & & \\ & & & & 1 \\ & & E_{d-1} & & \\ & -1 & & & \\ & & & & \end{pmatrix}, \quad \dot{s}_0 = \begin{pmatrix} & & & -p^{-1} & \\ & E_{d-1} & & & \\ p & & & & \\ & & & & \\ & & & & E_{d-1} \end{pmatrix}.$$

Then \dot{s}_i for $0 \leq i \leq d$ belongs to $\text{Sp}_{2d}(\mathbb{Q}_p) \subset G$ and normalizes T . Its image element $s_i = s_{\alpha_i}$ in $N(T)/ZT_0 = \widehat{W}$ is the reflection corresponding to α_i . The $s_0, s_1, \dots, s_{d-1}, s_d$ are Coxeter generators of $W_{\text{aff}} \subset \widehat{W}$ satisfying the relations (10). Put

$$\dot{u} = \begin{pmatrix} & E_d^* \\ pE_d^* & \end{pmatrix}.$$

Then \dot{u} belongs to $N(T)$ and normalizes I and I_0 . Its image element u in $N(T)/ZT_0$ satisfies the relations (11). In $N(T)$ we consider

$$\phi = (p \cdot \text{id}) \dot{s}_d \dot{s}_{d-1} \cdots \dot{s}_1 \dot{s}_0.$$

We may rewrite this as $\phi = (p \cdot \text{id}) \dot{s}_{\beta(1)} \cdots \dot{s}_{\beta(d+1)}$ where we put $\beta(i) = d+1-i$ for $1 \leq i \leq d+1$. For $a, b \in \mathbb{Z}_{\geq 0}$ with $0 \leq b < d+1$ we put

$$C^{(a(d+1)+b)} = \phi^a s_d \cdots s_{d-b+1} C = \phi^a s_{\beta(1)} \cdots s_{\beta(b)} C.$$

Define the homomorphism

$$\tau : \mathbb{Z}_p^\times \longrightarrow T_0, \quad x \mapsto \text{diag}(xE_d, E_d).$$

Lemma 4.1. *We have $\phi^d \in T$ and $\phi^d N_0 \phi^{-d} \subset N_0$. The sequence $C = C^{(0)}, C^{(1)}, C^{(2)}, \dots$ satisfies hypothesis (4). In particular we may define $\alpha^{(j)} \in \Phi^+$ for all $j \geq 0$.*

- (b) *For all $j \geq 0$ we have $\alpha^{(j)} \circ \tau = \text{id}_{\mathbb{Z}_p^\times}$.*
- (c) *We have $\tau(a)\phi = \phi\tau(a)$ for all $a \in \mathbb{Z}_p^\times$.*

PROOF: (a) This is Lemma 3.1 in practice. A matrix computation shows $\phi^d = (-1)^{d-1} \text{diag}(p^{d+1} E_d, p^{d-1} E_d) \in T$. The group N_α for $\alpha \in \Phi^+$ is generated by $\epsilon_{i,j+d} \epsilon_{j,i+d}$ if $\alpha = e_i + e_j$ with $1 \leq i < j \leq d$, by $\epsilon_{i,i+d}$ if $\alpha = 2e_i$ with $1 \leq i \leq d$, and by $\epsilon_{i,j} \epsilon_{i+d,j+d}^{-1}$ if $\alpha = e_i - e_j$ with $1 \leq i < j \leq d$. Using this we find

$$\phi^d N_0 \phi^{-d} = \prod_{\alpha \in \Phi^+} \phi^d (N_0 \cap N_\alpha) \phi^{-d} = \prod_{\alpha \in \Phi^+} (N_0 \cap N_\alpha)^{p^{m_\alpha}},$$

$$m_\alpha = \begin{cases} 2 & : & \alpha = e_i + e_j \text{ with } 1 \leq i < j \leq d \\ 2 & : & \alpha = 2e_i \text{ with } 1 \leq i \leq d \\ 0 & : & \text{all other } \alpha \in \Phi^+ \end{cases}$$

In particular we find $\phi^d N_0 \phi^{-d} \subset N_0$ and $[N_0 : \phi^d N_0 \phi^{-d}] = p^{d(d+1)}$. Let $n \in \mathbb{N}$. We have $\ell(\phi^n) \leq n(d+1)$ because the image of ϕ in \widehat{W} is a product of $d+1$ Coxeter generators. If $m \geq n$ is such that $m \in d\mathbb{N}$ we then have, on the other hand,

$$p^{\ell(\phi^m)} \geq [I_0 : (I_0 \cap \phi^m I \phi^{-m})] \geq [N_0 : (N_0 \cap \phi^m I_0 \phi^{-m})] = [N_0 : \phi^m N_0 \phi^{-m}] = p^{m(d+1)}.$$

Thus

$$\ell(\phi^n) \geq \ell(\phi^m) - \ell(\phi^{m-n}) \geq m(d+1) - (m-n)(d+1) = n(d+1),$$

hence $\ell(\phi^n) = n(d+1)$. We have shown that ϕ is power multiplicative and

$$[N_0 : \phi^d N_0 \phi^{-d}] = [N_0 : (N_0 \cap \phi^d N_0 \phi^{-d})] = [I_0 : (I_0 \cap \phi^d I_0 \phi^{-d})].$$

We get $I_0 = N_0 \cdot (I_0 \cap \phi^d I_0 \phi^{-d})$ and that hypothesis (4) holds true.

(b) As $\phi^d \in T$ we have $\{\alpha^{(j)} \mid j \geq 0\} = \{\alpha \in \Phi^+ \mid m_\alpha \neq 0\}$. This implies (b).

(c) Another matrix computation. □

As explained in subsection 3.2 we now obtain a functor $M \mapsto \mathbf{D}(\Theta_* \mathcal{V}_M)$ from $\text{Mod}^{\text{fin}}(\mathcal{H}(G, I_0))$ to the category of (φ^{d+1}, Γ) -modules over $\mathcal{O}_{\mathcal{E}}$. As explained in [2], to compute it we need to understand the intermediate objects $H_0(\overline{\mathfrak{X}}_+, \Theta_* \mathcal{V}_M)$, acted on by $[\mathfrak{N}_0, \varphi^{d+1}, \Gamma]$.

Let $\mathcal{H}(G, I_0)'_{\text{aff}, k}$ denote the k -sub algebra of $\mathcal{H}(G, I_0)_k$ generated by $\mathcal{H}(G, I_0)_{\text{aff}, k}$ together with $T_{p \cdot \text{id}} = T_{\bar{u}^2}$ and $T_{p \cdot \text{id}}^{-1} = T_{p^{-1} \cdot \text{id}}$.

Suppose we are given a character $\lambda : \overline{T} \rightarrow k^\times$, a subset $\mathcal{J} \subset S_\lambda$ and some $b \in k^\times$. Define the numbers $0 \leq k_i = k_i(\lambda, \mathcal{J}) \leq p-1$ as in subsection 3.3. The character $\chi_{\lambda, \mathcal{J}}$ of $\mathcal{H}(G, I_0)_{\text{aff}, k}$ extends uniquely to a character

$$\chi_{\lambda, \mathcal{J}, b} : \mathcal{H}(G, I_0)'_{\text{aff}, k} \longrightarrow k$$

which sends $T_{p \cdot \text{id}}$ to b (see the proof of [8] Proposition 3). Define the $\mathcal{H}(G, I_0)_k$ -module

$$M = M[\lambda, \mathcal{J}, b] = \mathcal{H}(G, I_0)_k \otimes_{\mathcal{H}(G, I_0)'_{\text{aff}, k}} k.e$$

where $k.e$ denotes the one dimensional k -vector space on the basis element e , endowed with the action of $\mathcal{H}(G, I_0)'_{\text{aff}, k}$ by the character $\chi_{\lambda, \mathcal{J}, b}$. As a k -vector space, M has dimension 2, a k -basis is e, f where we write $e = 1 \otimes e$ and $f = T_{\bar{u}} \otimes e$.

Definition: We call an $\mathcal{H}(G, I_0)_k$ -module *standard supersingular* if it is isomorphic with $M[\lambda, \mathcal{J}, b]$ for some λ, \mathcal{J}, b such that $k_\bullet \notin \{(0, \dots, 0), (p-1, \dots, p-1)\}$.

For $0 \leq j \leq d$ put $\tilde{j} = d - j$. Letting $\tilde{\beta} = (\cdot) \circ \beta$ we then have

$$\dot{u}\phi\dot{u}^{-1} = (p \cdot \text{id})\dot{s}_{\tilde{\beta}(1)} \cdots \dot{s}_{\tilde{\beta}(d+1)}.$$

Put $n_e = \sum_{i=0}^d k_{d-i}p^i = \sum_{i=0}^d k_{\beta(i+1)}p^i$ and $n_f = \sum_{i=0}^d k_i p^i = \sum_{i=0}^d k_{\tilde{\beta}(i+1)}p^i$. Put $\varrho = \prod_{i=0}^d (k_i!) = \prod_{i=0}^d (k_{\beta(i+1)}!) = \prod_{i=0}^d (k_{\tilde{\beta}(i+1)}!)$. Let $0 \leq s_e, s_f \leq p-2$ be such that $\lambda(\tau(x)) = x^{-s_e}$ and $\lambda(\dot{u}\tau(x)\dot{u}^{-1}) = x^{-s_f}$ for all $x \in \mathbb{F}_p^\times$.

Lemma 4.2. *The assignment $M[\lambda, \mathcal{J}, b] \mapsto (n_e, s_e, b\varrho^{-1})$ induces a bijection between the set of isomorphism classes of standard supersingular $\mathcal{H}(G, I_0)_k$ -modules and $\mathfrak{S}_C(d+1)$.*

PROOF: We have $\prod_{i=0}^d \alpha_i^\vee(x) = 1$ for all $x \in \mathbb{F}_p^\times$ (as can be seen e.g. from formula (12)). This implies

$$(13) \quad \sum_{i=0}^d k_i \equiv n_e \equiv n_f \equiv 0 \pmod{p-1}.$$

One can deduce from [8] Proposition 3 that for two sets of data λ, \mathcal{J}, b and $\lambda', \mathcal{J}', b'$ the $\mathcal{H}(G, I_0)_k$ -modules $M[\lambda, \mathcal{J}, b]$ and $M[\lambda', \mathcal{J}', b']$ are isomorphic if and only if $b = b'$ and the pair (λ, \mathcal{J}) is conjugate with the pair (λ', \mathcal{J}') by means of a power of \dot{u} , i.e. by means of $\dot{u}^0 = 1$ or $\dot{u}^1 = \dot{u}$. Conjugating (λ, \mathcal{J}) by \dot{u} has the effect of substituting k_{d-i} with k_i , for any i . The datum of the character λ is equivalent with the datum of s_e together with all the k_i taken modulo $(p-1)$ since the images of τ and all α_i^\vee together generate \overline{T} . Knowing the set \mathcal{J} is then equivalent with knowing the numbers k_i themselves (not just modulo $(p-1)$). Thus, our mapping is well defined and bijective. \square

Fix $M = M[\lambda, \mathcal{J}, b]$. Let $0 \leq j \leq d$. Normalize the homomorphism $\iota_{\alpha_j} : \text{SL}_2(\mathbb{Q}_p) \rightarrow G$ such that $\iota_{\alpha_j}(\dot{s}) = \dot{s}_j$ for $\dot{s} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Let $t_j = \iota_{\alpha_j}([\nu]) - 1 \in k[[\iota_{\alpha_j}\mathfrak{N}_0]] \subset k[[N_0]]$. Let F_j denote the codimension-1-face of C contained in the (affine) reflection hyperplane (in $A \subset X$) for s_j .

Lemma 4.3. *In $\mathcal{V}_M^X(F_j)$ we have $t_j^{k_j}\dot{s}_j e = k_j!e$ and $t_j^{k_{d-j}}\dot{s}_j f = k_{d-j}!f$ for all $0 \leq j \leq d$.*

PROOF: Let I_0^j denote the subgroup of G generated by I_0 and $\dot{s}_j I_0 \dot{s}_j^{-1}$. Let \overline{T}_0^j denote the maximal reductive (over \mathbb{F}_p) quotient of I_0^j . Then $\iota_{\alpha_j}(\text{SL}_2(\mathbb{Z}_p)) \subset I_0^j$ and ι_{α_j} induces an isomorphism between either the quotient $\overline{\mathcal{S}} = \text{SL}_2(\mathbb{F}_p)$ of $\text{SL}_2(\mathbb{Z}_p)$, or the quotient $\overline{\mathcal{S}} = \text{PSL}_2(\mathbb{F}_p)$ of $\text{SL}_2(\mathbb{Z}_p)$, with the quotient \overline{T}_0^j of I_0^j . The subgroup $\overline{\mathcal{U}}$ of $\overline{\mathcal{S}}$ generated

by (the image of) ν maps isomorphically to the image of I_0 in \overline{I}_0^j . Therefore we get an embedding of k -algebras

$$\begin{aligned}\mathcal{H}(\overline{\mathcal{S}}, \overline{\mathcal{U}})_k &= \text{End}_{k[\overline{\mathcal{S}}]}(\text{ind}_{\overline{\mathcal{U}}}^{\overline{\mathcal{S}}} \mathbf{1}_k)^{\text{op}} \cong \text{End}_{k[I_0^j]}(\text{ind}_{I_0^j}^{\overline{\mathcal{S}}} \mathbf{1}_k)^{\text{op}} \\ &\hookrightarrow \mathcal{H}(G, I_0)_{\text{aff}, k} \subset \mathcal{H}(G, I_0)_k.\end{aligned}$$

Let $\rho : \mathcal{H}(\overline{\mathcal{S}}, \overline{\mathcal{U}})_k \rightarrow k$ be the character defined by $\rho(T_{\dot{s}}) = -1$ if $k_j = p-1$ but $\rho(T_{\dot{s}}) = 0$ if $0 \leq k_j < p-1$, and by $\rho(T_{h(x)}) = x^{-k_j}$. We have a morphism of $\text{SL}_2(\mathbb{Z}_p)$ -representations (acting on $\mathcal{V}_M^X(F_j)$ through ι_{α_j})

$$(\text{ind}_{\overline{\mathcal{U}}}^{\overline{\mathcal{S}}} \mathbf{1}_k) \otimes_{\mathcal{H}(\overline{\mathcal{S}}, \overline{\mathcal{U}})_k} \rho \longrightarrow \mathcal{V}_M^X(F_j)$$

sending a basis element e of ρ to $e \in \mathcal{V}_M^X(F_j)$. It is injective since its restriction to the space $k.e$ of \mathcal{U} -invariants (see [2] Lemma 2.3) is injective. By [2] Lemma 2.5 we have $t^{k_j} \dot{s}e = k_j!e$ in $(\text{ind}_{\overline{\mathcal{U}}}^{\overline{\mathcal{S}}} \mathbf{1}_k) \otimes_{\mathcal{H}(\overline{\mathcal{S}}, \overline{\mathcal{U}})_k} \rho$ where $t = [\nu] - 1 \in k[\overline{\mathcal{U}}]$. We get $t_j^{k_j} \dot{s}_j e = k_j!e$ in $\mathcal{V}_M^X(F_j)$. A similar argument gives $t_j^{k_{d-j}} \dot{s}_j f = k_{d-j}!f$. For this notice that as the Hecke operator T_t for $t \in \overline{T}$ acts on $k.e$ through $\lambda(t^{-1})$, it acts on $k.f = k.T_{\dot{u}}e$ through $\lambda(\dot{u}t^{-1}\dot{u}^{-1})$ (the same computation as in formula (18) below), and that formula (9) implies $\lambda(\dot{u}\alpha_j^\vee(x)\dot{u}^{-1}) = x^{k_{d-j}}$. \square

Lemma 4.4. *For an appropriate choice of the isomorphism Θ we have, in $H_0(\overline{\mathfrak{X}}_+, \Theta_*\mathcal{V}_M)$,*

$$(14) \quad t^{n_e} \varphi^{d+1} e = \varrho b^{-1} e,$$

$$(15) \quad t^{n_f} \varphi^{d+1} f = \varrho b^{-1} f,$$

$$(16) \quad \gamma(x) e = x^{-s_e} e,$$

$$(17) \quad \gamma(x) f = x^{-s_f} f$$

for $x \in \mathbb{F}_p^\times$. The action of Γ_0 on $H_0(\overline{\mathfrak{X}}_+, \Theta_*\mathcal{V}_M)$ is trivial on the subspace M .

PROOF: The construction of Θ in [2] is based on the choice of elements $\nu_i \in N_0$ topologically generating the relevant pro- p subgroup 'at distance i ', for each $0 \leq i \leq d$. These ν_i can be taken to be the elements $y_{i-1} \cdots y_0 \iota_{d-i}(\nu) y_0^{-1} \cdots y_{i-1}^{-1}$. Their actions on the half-tree Y correspond to the actions of the ν^{p^i} on the half-tree \mathfrak{X}_+ .

We use the notations and the statements of Lemma 3.2, observing $\beta(i+1) = d-i$. For $0 \leq i \leq d$ we have $y_{i-1} \cdots y_0 F_{d-i} = C^{(i)} \cap C^{(i+1)}$ and $y_{i-1} \cdots y_0 C = C^{(i)}$. Thus $y_{i-1} \cdots y_0$ defines isomorphisms

$$\mathcal{V}_M^X(F_{d-i}) \cong \mathcal{V}_M^X(C^{(i)} \cap C^{(i+1)}) \quad \text{and} \quad \mathcal{V}_M^X(C) \cong \mathcal{V}_M^X(C^{(i)}).$$

But we have

$$\mathcal{V}_M^X(C^{(i)} \cap C^{(i+1)}) = \Theta_*\mathcal{V}_M(\mathfrak{v}_i) \quad \text{and} \quad \mathcal{V}_M^X(C^{(i)}) = \Theta_*\mathcal{V}_M(\mathfrak{e}_i).$$

It follows that, under the above isomorphisms, the action of t_{d-i} , resp. of s_{d-i} , on $\mathcal{V}_M^X(F_{d-i})$ becomes the action of $[\nu]^{p^i} - 1$, resp. of y_i , on $\Theta_* \mathcal{V}_M(\mathfrak{v}_i)$. Now as we are in characteristic p we have $t^{p^i} = ([\nu] - 1)^{p^i} = [\nu]^{p^i} - 1$. Applying this to the element e , resp. f , of $\mathcal{V}_M^X(C) \subset \mathcal{V}_M^X(F_{d-i})$, Lemma 4.3 tells us

$$(t^{p^i})^{k_{d-i}} y_i \cdots y_0 e = k_{d-i}! y_{i-1} \cdots y_0 e \quad \text{resp.} \quad (t^{p^i})^{k_i} y_i \cdots y_0 f = k_i! y_{i-1} \cdots y_0 f.$$

We compose these formulae for all $0 \leq i \leq d$ and finally recall that the central element $p \cdot \text{id}$ acts on M through the Hecke operator $T_{p^{-1}, \text{id}}$, i.e. by b^{-1} . We get formulae (15) and (14).

Next recall that the action of $\gamma(x)$ on $H_0(\overline{\mathfrak{X}}_+, \Theta_* \mathcal{V}_M)$ is given by that of $\tau(x) \in T$, i.e. by the Hecke operator $T_{\tau(x)^{-1}}$. We thus compute

$$\begin{aligned} \gamma(x)e &= T_{\tau(x)^{-1}}e = \lambda(\tau(x))e, \\ (18) \quad \gamma(x)f &= T_{\tau(x)^{-1}}T_{\dot{u}}e = T_{\dot{u}\tau(x)^{-1}}e = T_{\dot{u}}T_{\dot{u}\tau(x)^{-1}\dot{u}^{-1}}e = T_{\dot{u}}\lambda(\dot{u}\tau(x)\dot{u}^{-1})e = \lambda(\dot{u}\tau(x)\dot{u}^{-1})f \end{aligned}$$

and obtain formulae (16) and (17). \square

Corollary 4.5. *The étale (φ^{d+1}, Γ) -module $\mathbf{D}(\Theta_* \mathcal{V}_M)$ over $k_{\mathcal{E}}$ associated with $H_0(\overline{\mathfrak{X}}_+, \Theta_* \mathcal{V}_M)$ admits a $k_{\mathcal{E}}$ -basis g_e, g_f such that*

$$\begin{aligned} \varphi^{d+1}g_e &= b\varrho^{-1}t^{n_e+1-p^{d+1}}g_e \\ \varphi^{d+1}g_f &= b\varrho^{-1}t^{n_f+1-p^{d+1}}g_f \\ \gamma(x)g_e - x^{s_e}g_e &\in t \cdot k_{\mathcal{E}}^+ \cdot g_e \\ \gamma(x)g_f - x^{s_f}g_f &\in t \cdot k_{\mathcal{E}}^+ \cdot g_f. \end{aligned}$$

PROOF: This follows from Lemma 4.4 as explained in [2] Lemma 6.4. \square

Corollary 4.6. *The functor $M \mapsto \mathbf{D}(\Theta_* \mathcal{V}_M)$ induces a bijection between*

- (a) *the set of isomorphism classes of standard supersingular $\mathcal{H}(G, I_0)_k$ -modules, and*
- (b) *the set of isomorphism classes of C -symmetric étale (φ^{d+1}, Γ) -modules over $k_{\mathcal{E}}$.*

PROOF: For $x \in \mathbb{F}_p^\times$ we have

$$\tau(x) \cdot \dot{u}\tau^{-1}(x)\dot{u}^{-1} = \text{diag}(xE_d, x^{-1}E_d) = \left(\sum_{i=0}^d (i+1)\alpha_i^\vee \right)(x)$$

in \overline{T} . Applying λ and observing $\sum_{i=0}^d k_i \equiv 0$ modulo $(p-1)$ we get

$$x^{s_f - s_e} = \lambda\left(\left(\sum_{i=0}^d (i+1)\alpha_i^\vee\right)(x)\right) = x^{\sum_{i=0}^d ik_i}$$

and hence $s_f - s_e \equiv \sum_{i=0}^d ik_i$ modulo $(p-1)$. Therefore

$$\mathbf{D}(\Theta_*\mathcal{V}_M) = \mathbf{D}(\Theta_*\mathcal{V}_M)_1 \oplus \mathbf{D}(\Theta_*\mathcal{V}_M)_2$$

with $\mathbf{D}(\Theta_*\mathcal{V}_M)_1 = k_{\mathcal{E}}g_e$ and $\mathbf{D}(\Theta_*\mathcal{V}_M)_2 = k_{\mathcal{E}}g_f$ is a direct sum decomposition of $\mathbf{D}(\Theta_*\mathcal{V}_M)$ as an étale (φ^{d+1}, Γ) -module over $k_{\mathcal{E}}$, and by Corollary (4.5) this decomposition identifies $\mathbf{D}(\Theta_*\mathcal{V}_M)$ as being C -symmetric (recall in particular that $n_e = \sum_{i=0}^d k_{d-i}p^i$ and $n_f = \sum_{i=0}^d k_i p^i$). Composing this assignment $M \mapsto \mathbf{D}(\Theta_*\mathcal{V}_M)$ (from standard supersingular $\mathcal{H}(G, I_0)_k$ -modules to C -symmetric étale (φ^{d+1}, Γ) -modules) with the bijection of Lemma 2.5 (i) (from C -symmetric étale (φ^{d+1}, Γ) -modules to $\mathfrak{S}_C(d+1)$) we get the bijection of Lemma 4.2 (from standard supersingular $\mathcal{H}(G, I_0)_k$ -modules to $\mathfrak{S}_C(d+1)$). Thus, $M \mapsto \mathbf{D}(\Theta_*\mathcal{V}_M)$ is a bijection as stated. \square

Remark: Consider the subgroup $G' = \mathrm{Sp}_{2d}(\mathbb{Q}_p)$ of G . If we replace the above τ by $\tau : \mathbb{Z}_p^\times \rightarrow T_0$, $x \mapsto \mathrm{diag}(xE_d, x^{-1}E_d)$ and if we replace the above ϕ by $\phi = \dot{s}_d \dot{s}_{d-1} \cdots \dot{s}_1 \dot{s}_0$ then everything in fact happens inside G' . We then have $\alpha^{(j)} \circ \tau = \mathrm{id}_{\mathbb{Z}_p^\times}^2$ for all $j \geq 0$. Let $\mathrm{Mod}_0^{\mathrm{fin}}\mathcal{H}(G', G' \cap I_0)$ denote the category of finite- \mathfrak{o} -length $\mathcal{H}(G', G' \cap I_0)$ -modules on which $\tau(-1)$ (i.e. $T_{\tau(-1)} = T_{\tau(-1)-1}$) acts trivially. For $M \in \mathrm{Mod}_0^{\mathrm{fin}}\mathcal{H}(G', G' \cap I_0)$ we obtain an action of $[\mathfrak{N}_0, \varphi^{d+1}, \Gamma^2]$ on $H_0(\bar{\mathfrak{X}}_+, \Theta_*\mathcal{V}_M)$, where $\Gamma^2 = \{\gamma^2 \mid \gamma \in \Gamma\} \subset \Gamma$. Correspondingly, following [2] (as a slight variation from what we explained in subsection 3.2), we obtain a functor from $\mathrm{Mod}_0^{\mathrm{fin}}\mathcal{H}(G', G' \cap I_0)$ to the category of $(\varphi^{d+1}, \Gamma^2)$ -modules over $\mathcal{O}_{\mathcal{E}}$.

Remark: In the case $d = 2$ one may also work with $\phi = (p \cdot \mathrm{id})\dot{s}_2\dot{s}_1\dot{s}_2\dot{u}$. Its square is $p \cdot \mathrm{id}$ times the square of the $\phi = (p \cdot \mathrm{id})\dot{s}_2\dot{s}_1\dot{s}_0$ used above.

4.2 Affine root system \tilde{B}_d

Assume $d \geq 3$. Here W_{aff} is the Coxeter group with generators s_0, s_1, \dots, s_d and relations

$$(19) \quad (s_{d-1}s_d)^4 = 1 \quad \text{and} \quad (s_0s_2)^3 = (s_{i-1}s_i)^3 = 1 \quad \text{for } 2 \leq i \leq d-1$$

and moreover $(s_i s_j)^2 = 1$ for all other pairs $i < j$, and $s_i^2 = 1$ for all i . In the extended affine Weyl group \widehat{W} we find (cf. [4]) an element u of length 0 with

$$(20) \quad u^2 = 1 \quad \text{and} \quad us_0u = s_1 \quad \text{and} \quad us_iu = s_i \quad \text{for } 2 \leq i \leq d.$$

(We have $\widehat{W} = W_{\mathrm{aff}} \rtimes W_{\Omega}$ with the two-element subgroup $W_{\Omega} = \{1, u\}$.) Let

$$\tilde{S}_d = \begin{pmatrix} S_d & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_{2d+1}(\mathbb{Q}_p)$$

and consider the special orthogonal group

$$G = \mathrm{SO}_{2d+1}(\mathbb{Q}_p) = \{A \in \mathrm{SL}_{2d+1}(\mathbb{Q}_p) \mid {}^T A \tilde{S}_d A = \tilde{S}_d\}.$$

Let T denote the maximal torus consisting of all diagonal matrices in G . For $1 \leq i \leq d$ let

$$e_i : T \longrightarrow \mathbb{Q}_p^\times, \quad \mathrm{diag}(x_1, \dots, x_d, x_1^{-1}, \dots, x_d^{-1}, 1) \mapsto x_i.$$

Then (in *additive* notation) $\Phi = \{\pm e_i \pm e_j \mid i \neq j\} \cup \{\pm e_i\}$ is the root system of G with respect to T . It is of type B_d . We choose the positive system $\Phi^+ = \{e_i \pm e_j \mid i < j\} \cup \{e_i \mid 1 \leq i \leq d\}$ with corresponding set of simple roots $\Delta = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \dots, \alpha_{d-1} = e_{d-1} - e_d, \alpha_d = e_d\}$. The negative of the highest root is $\alpha_0 = -e_1 - e_2$. For $0 \leq i \leq d$ we have the following explicit formula for $\alpha_i^\vee = (\alpha_i)^\vee$:

$$(21) \quad \alpha_i^\vee(x) = \begin{cases} \mathrm{diag}(x^{-1}, x^{-1}, E_{d-2}, x, x, E_{d-2}, 1) & : \quad i = 0 \\ \mathrm{diag}(E_{i-1}, x, x^{-1}, E_{d-i-1}, E_{i-1}, x^{-1}, x, E_{d-i-1}, 1) & : \quad 1 \leq i \leq d-1 \\ \mathrm{diag}(E_{d-1}, x^2, E_{d-1}, x^{-2}, 1) & : \quad i = d \end{cases}$$

Remark: For roots $\alpha \in \Phi$ of the form $\alpha = \pm e_i \pm e_j$ the homomorphism $\iota_\alpha : \mathrm{SL}_2 \rightarrow \mathrm{SO}_{2d+1}$ is injective. For roots $\alpha \in \Phi$ of the form $\alpha = \pm e_i$ the homomorphism $\iota_\alpha : \mathrm{SL}_2 \rightarrow \mathrm{SO}_{2d+1}$ induces an embedding $\mathrm{PSL}_2 \rightarrow \mathrm{SO}_{2d+1}$.

For $\alpha \in \Phi$ let N_α^0 be the subgroup of the corresponding root subgroup N_α of G all of whose elements belong to $\mathrm{SL}_{2d+1}(\mathbb{Z}_p)$. Let I_0 denote the pro- p -Iwahori subgroup generated by the N_α^0 for all $\alpha \in \Phi^+$, by the $(N_\alpha^0)^p$ for all $\alpha \in \Phi^- = \Phi - \Phi^+$, and by the maximal pro- p -subgroup of T_0 . Let I denote the Iwahori subgroup of G containing I_0 . Let N_0 be the subgroup of G generated by all N_α^0 for $\alpha \in \Phi^+$.

For $1 \leq i \leq d-1$ define the block diagonal matrix

$$\dot{s}_i = \mathrm{diag}(E_{i-1}, S_1, E_{d-i-1}, E_{i-1}, S_1, E_{d-i-1}, 1)$$

and furthermore

$$\dot{s}_d = \begin{pmatrix} E_{d-1} & & & & & \\ & & & & 1 & \\ & & E_{d-1} & & & \\ & 1 & & & & \\ & & & & & -1 \end{pmatrix}.$$

Define

$$\dot{u} = \begin{pmatrix} & & p^{-1} & & & \\ & E_{d-1} & & & & \\ p & & & & & \\ & & & E_{d-1} & & \\ & & & & & -1 \end{pmatrix}$$

and $\dot{s}_0 = \dot{u}\dot{s}_1\dot{u}$. Then \dot{s}_i for $0 \leq i \leq d$ belongs to G and normalizes T . Its image element $s_i = s_{\alpha_i}$ in $N(T)/T_0 = \widehat{W}$ is the reflection corresponding to α_i . The $s_0, s_1, \dots, s_{d-1}, s_d$ are Coxeter generators of $W_{\text{aff}} \subset \widehat{W}$ satisfying the relations (19). Also \dot{u} belongs to G ; it normalizes T , I and I_0 . Its image element u in $N(T)/T_0 = \widehat{W}$ satisfies the relations (20). In $N(T)$ we consider the element

$$(22) \quad \phi = \dot{s}_1 \dot{s}_2 \cdots \dot{s}_{d-1} \dot{s}_d \dot{s}_{d-1} \cdots \dot{s}_2 \dot{s}_0.$$

We may rewrite this as $\phi = \dot{s}_{\beta(1)} \cdots \dot{s}_{\beta(2d-1)}$ where we put $\beta(i) = i$ for $1 \leq i \leq d$ and $\beta(i) = 2d - i$ for $d \leq i \leq 2d - 2$ and $\beta(2d - 1) = 0$. We put

$$C^{(a(2d-1)+b)} = \phi^a s_{\beta(1)} \cdots s_{\beta(b)} C$$

for $a, b \in \mathbb{Z}_{\geq 0}$ with $0 \leq b < 2d - 1$. Define the homomorphism

$$\tau : \mathbb{Z}_p^\times \longrightarrow T_0, \quad x \mapsto \text{diag}(x, E_{d-1}, x^{-1}, E_{d-1}, 1).$$

Lemma 4.7. *We have $\phi^2 \in T$ and $\phi^2 N_0 \phi^{-2} \subset N_0$. The sequence $C = C^{(0)}, C^{(1)}, C^{(2)}, \dots$ satisfies hypothesis (4). In particular we may define $\alpha^{(j)} \in \Phi^+$ for all $j \geq 0$.*

(b) *For any $j \geq 0$ we have $\alpha^{(j)} \circ \tau = \text{id}_{\mathbb{Z}_p^\times}$.*

(c) *We have $\tau(a)\phi = \phi\tau(a)$ for all $a \in \mathbb{Z}_p^\times$.*

PROOF: (a) A matrix computation shows $\phi^2 = \text{diag}(p^2, E_{d-1}, p^{-2}, E_{d-1}, 1)$. The group N_α for $\alpha \in \Phi^+$ is generated by $\epsilon_{i,j+d}\epsilon_{j,i+d}^{-1}$ if $\alpha = e_i + e_j$ with $1 \leq i < j \leq d$, by $\epsilon_{i,j}\epsilon_{j+d,i+d}^{-1}$ if $\alpha = e_i - e_j$ with $1 \leq i < j \leq d$, and by $\epsilon_{i,2d+1}\epsilon_{2d+1,i+d}^{-1}$ if $\alpha = e_i$ with $1 \leq i \leq d$. Using this we find

$$\phi^2 N_0 \phi^{-2} = \prod_{\alpha \in \Phi^+} \phi^2 (N_0 \cap N_\alpha) \phi^{-2} = \prod_{\alpha \in \Phi^+} (N_0 \cap N_\alpha)^{p^{m_\alpha}},$$

$$m_\alpha = \begin{cases} 2 & : & \alpha = e_1 - e_i \text{ with } 1 < i \\ 2 & : & \alpha = e_1 + e_i \text{ with } 1 < i \\ 2 & : & \alpha = e_1 \\ 0 & : & \text{all other } \alpha \in \Phi^+ \end{cases}$$

In particular we find $\phi^2 N_0 \phi^{-2} \subset N_0$ and $[N_0 : \phi^2 N_0 \phi^{-2}] = p^{2(2d-1)}$. On the other hand, the image of ϕ in \widehat{W} is a product of $2d - 1$ Coxeter generators. Arguing as in the proof of Lemma 4.1 we combine these facts to obtain $\ell(\phi) = 2d - 1$ and that ϕ is power multiplicative. We obtain that hypothesis (4) holds true, again by the same reasoning as in Lemma 4.1.

(b) As $\phi^2 \in T$ we have $\{\alpha^{(j)} \mid j \geq 0\} = \{\alpha \in \Phi^+ \mid m_\alpha \neq 0\}$. This implies (b).

(c) Another matrix computation. □

As explained in subsection 3.2 we now obtain a functor from $\text{Mod}^{\text{fin}}(\mathcal{H}(G, I_0))$ to the category of (φ^{2d-1}, Γ) -modules over $\mathcal{O}_{\mathcal{E}}$.

Suppose we are given a character $\lambda : \overline{T} \rightarrow k^\times$ and a subset $\mathcal{J} \subset S_\lambda$. Define the numbers $0 \leq k_i = k_i(\lambda, \mathcal{J}) \leq p-1$ as in subsection 3.3. Notice that k_d is necessarily even since in $\alpha_d^\vee(x)$ (formula (21)) the entry x only appears squared. Define the $\mathcal{H}(G, I_0)_k$ -module

$$M = M[\lambda, \mathcal{J}] = \mathcal{H}(G, I_0)_k \otimes_{\mathcal{H}(G, I_0)_{\text{aff}, k}} k.e$$

where $k.e$ denotes the one dimensional k -vector space on the basis element e , endowed with the action of $\mathcal{H}(G, I_0)_{\text{aff}, k}$ by the character $\chi_{\lambda, \mathcal{J}}$. As a k -vector space, M has dimension 2, a k -basis is e, f where we write $e = 1 \otimes e$ and $f = T_{\dot{u}} \otimes e$.

Definition: We call an $\mathcal{H}(G, I_0)_k$ -module *standard supersingular* if it is isomorphic with $M[\lambda, \mathcal{J}]$ for some λ, \mathcal{J} such that $k_\bullet \notin \{(0, \dots, 0), (p-1, \dots, p-1)\}$.

A *packet* of standard supersingular $\mathcal{H}(G, I_0)_k$ -modules is a set of isomorphism classes of standard supersingular $\mathcal{H}(G, I_0)_k$ -modules all of which give rise to the same \mathcal{J} and the same $k_\bullet = (k_i)_i$.

Remark: Let \overline{T}' denote the subgroup of \overline{T} generated by the $\alpha_i^\vee(\mathbb{F}_p^\times)$ for all $0 \leq i \leq d$. Formula (21) implies $[\overline{T} : \overline{T}'] = 2$. Two supersingular $\mathcal{H}(G, I_0)_k$ -modules $M[\lambda, \mathcal{J}]$ and $M[\lambda', \mathcal{J}']$ belong to the same packet if and only if $\mathcal{J} = \mathcal{J}'$ and if the restrictions of λ and λ' to \overline{T}' coincide.

For $2 \leq j \leq d$ we put $\tilde{j} = j$, furthermore we put $\tilde{0} = 1$ and $\tilde{1} = 0$. Letting $\tilde{\beta} = (\cdot) \circ \beta$ we then have

$$\dot{u}\phi\dot{u}^{-1} = \dot{s}_{\tilde{\beta}(1)} \cdots \dot{s}_{\tilde{\beta}(2d-1)}.$$

Put $n_e = \sum_{i=0}^{2d-2} k_{\beta(i+1)} p^i$ and $n_f = \sum_{i=0}^{2d-2} k_{\tilde{\beta}(i+1)} p^i$. Put $\varrho = k_0!k_1!k_d! \prod_{i=2}^{d-1} (k_i!)^2 = \prod_{i=0}^{2d-2} (k_{\beta(i+1)}!) = \prod_{i=0}^{2d-2} (k_{\tilde{\beta}(i+1)}!)$. Let $0 \leq s_e, s_f \leq p-2$ be such that $\lambda(\tau(x)) = x^{-s_e}$ and $\lambda(\dot{u}\tau(x)\dot{u}^{-1}) = x^{-s_f}$ for all $x \in \mathbb{F}_p^\times$.

Lemma 4.8. *The assignment $M[\lambda, \mathcal{J}] \mapsto (n_e, s_e)$ induces a bijection between the set of packets of standard supersingular $\mathcal{H}(G, I_0)_k$ -modules and $\mathfrak{S}_B(2d-1)$.*

PROOF: We have $\alpha_0^\vee(x)\alpha_1^\vee(x)\alpha_d^\vee(x) \prod_{i=2}^{d-1} (\alpha_i^\vee)^2(x) = 1$ for all $x \in \mathbb{F}_p^\times$ (as can be seen e.g. from formula (21)). This implies

$$(23) \quad k_0 + k_1 + k_d + 2 \sum_{i=2}^{d-1} k_i \equiv n_e \equiv n_f \equiv 0 \pmod{p-1}.$$

We further proceed exactly as in the proof of Lemma 4.2. \square

Lemma 4.9. *Let $M = M[\lambda, \mathcal{J}]$ for some λ, \mathcal{J} . For an appropriate choice of the isomorphism Θ we have, in $H_0(\overline{\mathfrak{X}}_+, \Theta_* \mathcal{V}_M)$,*

$$(24) \quad t^{n_e} \varphi^{2d-1} e = \varrho e,$$

$$(25) \quad t^{n_f} \varphi^{2d-1} f = \varrho f,$$

$$(26) \quad \gamma(x) e = x^{-s_e} e,$$

$$(27) \quad \gamma(x) f = x^{-s_f} f$$

for $x \in \mathbb{F}_p^\times$. The action of Γ_0 on $H_0(\overline{\mathfrak{X}}_+, \Theta_* \mathcal{V}_M)$ is trivial on the subspace M .

PROOF: As in Lemma 4.4. \square

Corollary 4.10. *The étale (φ^{2d-1}, Γ) -module over $k_{\mathcal{E}}$ associated with $H_0(\overline{\mathfrak{X}}_+, \Theta_* \mathcal{V}_M)$ admits a $k_{\mathcal{E}}$ -basis g_e, g_f such that*

$$\varphi^{2d-1} g_e = \varrho^{-1} t^{n_e+1-p^{2d-1}} g_e$$

$$\varphi^{2d-1} g_f = \varrho^{-1} t^{n_f+1-p^{2d-1}} g_f$$

$$\gamma(x) g_e - x^{s_e} g_e \in t \cdot k_{\mathcal{E}}^+ \cdot g_e$$

$$\gamma(x) g_f - x^{s_f} g_f \in t \cdot k_{\mathcal{E}}^+ \cdot g_f.$$

PROOF: This follows from Lemma 4.9 as explained in [2] Lemma 6.4. \square

Corollary 4.11. *The functor $M \mapsto \mathbf{D}(\Theta_* \mathcal{V}_M)$ induces a bijection between*

(a) *the set of packets of standard supersingular $\mathcal{H}(G, I_0)_k$ -modules, and*

(b) *the set of isomorphism classes of B -symmetric étale (φ^{2d-1}, Γ) -modules \mathbf{D} over $k_{\mathcal{E}}$.*

PROOF: For $x \in \mathbb{F}_p^\times$ we compute

$$\tau(x) \cdot \dot{u} \tau^{-1}(x) \dot{u}^{-1} = \text{diag}(x^2, E_{d-1}, x^{-2}, E_{d-1}, 1) = (\alpha_1^\vee - \alpha_0^\vee)(x)$$

in \overline{T} . Application of λ gives $x^{s_f-s_e} = x^{k_1-k_0}$ and hence $s_f - s_e \equiv k_1 - k_0 = k_{\beta(1)} - k_{\beta(2d-1)}$ modulo $(p-1)$. The required symmetry in the p -adic digits of n_e, n_f is due to the corresponding symmetry of the function β . Thus, $\mathbf{D}(\Theta_* \mathcal{V}_M)$ is a B -symmetric étale (φ^{2d-1}, Γ) -module.

One checks that composing $M \mapsto \mathbf{D}(\Theta_* \mathcal{V}_M)$ with the map of Lemma 2.5 (ii) gives the map of Lemma 4.8. As the maps in Lemmata 4.8 and 2.5 (ii) are bijective we obtain our result. \square

4.3 Affine root system \tilde{D}_d

Assume $d \geq 4$. Here W_{aff} is the Coxeter group with generators s_0, s_1, \dots, s_d and relations

$$(28) \quad (s_{d-2}s_d)^3 = (s_0s_2)^3 = (s_{i-1}s_i)^3 = 1 \quad \text{for } 2 \leq i \leq d-1$$

and moreover $(s_i s_j)^2 = 1$ for all other pairs $i < j$, and $s_i^2 = 1$ for all i . In the extended affine Weyl group \widehat{W} we find (cf. [4]) an element u of length 0 with

$$(29) \quad \begin{aligned} u^2 &= 1 & \text{and} & & us_0u &= s_1, & us_1u &= s_0, & us_{d-1}u &= s_d, & us_du &= s_{d-1} \\ & & & & us_iu &= s_i & \text{for } 2 \leq i \leq d-2. \end{aligned}$$

(We have $\widehat{W} = W_{\text{aff}} \rtimes W_\Omega$ with a four-element subgroup W_Ω , with $u \in W_\Omega$ of order 2.) Consider the general orthogonal group

$$\text{GO}_{2d}(\mathbb{Q}_p) = \{A \in \text{GL}_{2d}(\mathbb{Q}_p) \mid {}^T A S_d A = \kappa(A) S_d \text{ for some } \kappa(A) \in \mathbb{Q}_p^\times\}.$$

It contains the special orthogonal group

$$\text{SO}_{2d}(\mathbb{Q}_p) = \{A \in \text{SL}_{2d}(\mathbb{Q}_p) \mid {}^T A S_d A = S_d\}.$$

Let $G = \text{GSO}_{2d}(\mathbb{Q}_p)$ be the subgroup of $\text{GO}_{2d}(\mathbb{Q}_p)$ generated by $\text{SO}_{2d}(\mathbb{Q}_p)$ and by all $\text{diag}(xE_d, E_d)$ with $x \in \mathbb{Q}_p^\times$; it is of index 2 in $\text{GO}_{2d}(\mathbb{Q}_p)$.[‡]

Let T be the maximal torus consisting of all diagonal matrices in G . For $1 \leq i \leq d$ let

$$e_i : T \cap \text{SL}_{2d}(\mathbb{Q}_p) \longrightarrow \mathbb{Q}_p^\times, \quad \text{diag}(x_1, \dots, x_{2d}) \mapsto x_i.$$

For $1 \leq i, j \leq d$ and $\epsilon_1, \epsilon_2 \in \{\pm 1\}$ we thus obtain characters (using additive notation as usual) $\epsilon_1 e_i + \epsilon_2 e_j : T \cap \text{SL}_{2d}(\mathbb{Q}_p) \longrightarrow \mathbb{Q}_p^\times$. We extend these latter ones to T by setting

$$\epsilon_1 e_i + \epsilon_2 e_j : T \longrightarrow \mathbb{Q}_p^\times, \quad A = \text{diag}(x_1, \dots, x_{2d}) \mapsto x_i^{\epsilon_1} x_j^{\epsilon_2} \kappa(A)^{\frac{-\epsilon_1 - \epsilon_2}{2}}.$$

Then $\Phi = \{\pm e_i \pm e_j \mid i \neq j\}$ is the root system of G with respect to T . It is of type D_d . Choose the positive system $\Phi^+ = \{e_i \pm e_j \mid i < j\}$ with corresponding set of simple roots $\Delta = \{\alpha_1 = e_1 - e_2, \alpha_2 = e_2 - e_3, \dots, \alpha_{d-1} = e_{d-1} - e_d, \alpha_d = e_{d-1} + e_d\}$. The negative of the highest root is $\alpha_0 = -e_1 - e_2$. For $0 \leq i \leq d$ we have the following explicit formula for $\alpha_i^\vee = (\alpha_i)^\vee$:

$$(30) \quad \alpha_i^\vee(x) = \begin{cases} \text{diag}(x^{-1}, x^{-1}, E_{d-2}, x, x, E_{d-2}) & : & i = 0 \\ \text{diag}(E_{i-1}, x, x^{-1}, E_{d-i-1}, E_{i-1}, x^{-1}, x, E_{d-i-1}) & : & 1 \leq i \leq d-1 \\ \text{diag}(E_{d-2}, x, x, E_{d-2}, x^{-1}, x^{-1}) & : & i = d \end{cases}$$

[‡]With the obvious definitions, $\text{GO}_{2d}(\mathbb{Q}_p)$ is the group of \mathbb{Q}_p -valued points of an algebraic group GO_{2d} , whereas G is the group of \mathbb{Q}_p -valued points of the connected component GSO_{2d} of GO_{2d} ; this GSO_{2d} has connected center.

For $\alpha \in \Phi$ let N_α^0 be the subgroup of the corresponding root subgroup N_α of G all of whose elements belong to $\mathrm{SL}_{2d}(\mathbb{Z}_p)$. Let I_0 denote the pro- p -Iwahori subgroup generated by the N_α^0 for all $\alpha \in \Phi^+$, by the $(N_\alpha^0)^p$ for all $\alpha \in \Phi^- = \Phi - \Phi^+$, and by the maximal pro- p -subgroup of T_0 . Let I denote the Iwahori subgroup of G containing I_0 . Let N_0 be the subgroup of G generated by all N_α^0 for $\alpha \in \Phi^+$.

For $1 \leq i \leq d-1$ define the block diagonal matrix

$$\dot{s}_i = \mathrm{diag}(E_{i-1}, S_1, E_{d-i-1}, E_{i-1}, S_1, E_{d-i-1}) = \mathrm{diag}(E_{i-1}, S_1, E_{d-2}, S_1, E_{d-i-1}).$$

Put

$$\dot{u} = \begin{pmatrix} & & p^{-1} & & \\ & E_{d-2} & & & \\ & & & & 1 \\ p & & & & \\ & & & E_{d-2} & \\ & & 1 & & \end{pmatrix}$$

and $\dot{s}_0 = \dot{u}\dot{s}_1\dot{u}$ and $\dot{s}_d = \dot{u}\dot{s}_{d-1}\dot{u}$. Then \dot{s}_i for $0 \leq i \leq d$ belongs to $\mathrm{SO}_{2d}(\mathbb{Q}_p) \subset G$ and normalizes T . Its image element $s_i = s_{\alpha_i}$ in $N(T)/ZT_0 = \widehat{W}$ is the reflection corresponding to α_i . The $s_0, s_1, \dots, s_{d-1}, s_d$ are Coxeter generators of $W_{\mathrm{aff}} \subset \widehat{W}$ satisfying the relations (28). Also \dot{u} belongs to $\mathrm{SO}_{2d}(\mathbb{Q}_p) \subset G$; it normalizes T , I and I_0 . Its image element u in $N(T)/ZT_0 = \widehat{W}$ satisfies the relations (29). In $N(T)$ we consider

$$\begin{aligned} \phi &= (p \cdot \mathrm{id})\dot{s}_{d-1}\dot{s}_{d-2} \cdots \dot{s}_2\dot{s}_1\dot{s}_d\dot{s}_{d-2}\dot{s}_{d-3} \cdots \dot{s}_3\dot{s}_2\dot{s}_0 && \text{if } d \text{ is even,} \\ \phi &= (p^2 \cdot \mathrm{id})\dot{s}_{d-1}\dot{s}_{d-2} \cdots \dot{s}_2\dot{s}_1\dot{s}_d\dot{s}_{d-2}\dot{s}_{d-3} \cdots \dot{s}_3\dot{s}_2\dot{s}_0 && \text{if } d \text{ is odd.} \end{aligned}$$

(The reason for our distinction according to the parity of d is that the center Z of G is the subgroup of G generated by $T_0 \cap Z$ and $p \cdot \mathrm{id}$ if d is even, resp. by $T_0 \cap Z$ and $p^2 \cdot \mathrm{id}$ if d is odd.) We may rewrite this as $\phi = (p \cdot \mathrm{id})\dot{s}_{\beta(1)} \cdots \dot{s}_{\beta(2d-2)}$ if d is even, resp. $\phi = (p^2 \cdot \mathrm{id})\dot{s}_{\beta(1)} \cdots \dot{s}_{\beta(2d-2)}$ if d is odd, where $\beta(i) = d-i$ for $1 \leq i \leq d-1$, $\beta(d) = d$, $\beta(i) = 2d-1-i$ for $d+1 \leq i \leq 2d-3$ and $\beta(2d-2) = 0$. We put

$$C^{(a(2d-2)+b)} = \phi^a s_{\beta(1)} \cdots s_{\beta(b)} C$$

for $a, b \in \mathbb{Z}_{\geq 0}$ with $0 \leq b < 2d-2$. Define the homomorphism

$$\tau : \mathbb{Z}_p^\times \longrightarrow T_0, \quad x \mapsto \mathrm{diag}(xE_{d-1}, E_d, x).$$

Lemma 4.12. *We have $\phi^d \in T$ and $\phi^d N_0 \phi^{-d} \subset N_0$. The sequence $C = C^{(0)}, C^{(1)}, C^{(2)}, \dots$ satisfies hypothesis (4). In particular we may define $\alpha^{(j)} \in \Phi^+$ for all $j \geq 0$.*

(b) *For any $j \geq 0$ we have $\alpha^{(j)} \circ \tau = \mathrm{id}_{\mathbb{Z}_p^\times}$.*

(c) *We have $\tau(a)\phi = \phi\tau(a)$ for all $a \in \mathbb{Z}_p^\times$.*

PROOF: (a) A matrix computation shows $\phi^d = \text{diag}(p^{d+2}E_{d-1}, p^{d-2}E_d, p^{d+2})$ if d is even, and $\phi^d = \text{diag}(p^{2d+2}E_{d-1}, p^{2d-2}E_d, p^{2d+2})$ if d is odd. The group N_α for $\alpha \in \Phi^+$ is generated by $\epsilon_{i,j+d}\epsilon_{j,i+d}^{-1}$ if $\alpha = e_i + e_j$ with $1 \leq i < j \leq d$, and by $\epsilon_{i,j}\epsilon_{j+d,i+d}^{-1}$ if $\alpha = e_i - e_j$ with $1 \leq i < j \leq d$. Using this we find

$$\phi^d N_0 \phi^{-d} = \prod_{\alpha \in \Phi^+} \phi^d (N_0 \cap N_\alpha) \phi^{-d} = \prod_{\alpha \in \Phi^+} (N_0 \cap N_\alpha)^{p^{m_\alpha}},$$

$$m_\alpha = \begin{cases} 4 & : & \alpha = e_i + e_j \text{ with } 1 \leq i < j < d \\ 4 & : & \alpha = e_i - e_d \text{ with } 1 \leq i < d \\ 0 & : & \text{all other } \alpha \in \Phi^+ \end{cases}$$

In particular we find $\phi^d N_0 \phi^{-d} \subset N_0$ and $[N_0 : \phi^d N_0 \phi^{-d}] = p^{2d(d-1)}$. On the other hand, the image of ϕ in \widehat{W} is a product of $2d-2$ Coxeter generators and of an element of length 0. Arguing as in the proof of Lemma 4.1 we combine these facts to obtain $\ell(\phi) = 2d-2$ and that ϕ is power multiplicative. We obtain that hypothesis (4) holds true, again by the same reasoning as in Lemma 4.1.

(b) As $\phi^d \in T$ we have $\{\alpha^{(j)} \mid j \geq 0\} = \{\alpha \in \Phi^+ \mid m_\alpha \neq 0\}$. This implies (b).

(c) Another matrix computation. □

As explained in subsection 3.2 we now obtain a functor from $\text{Mod}^{\text{fin}}(\mathcal{H}(G, I_0))$ to the category of (φ^{2d-2}, Γ) -modules over $\mathcal{O}_\mathcal{E}$. Consider the elements

$$\dot{\omega} = \begin{pmatrix} & E_d^* \\ pE_d^* & \end{pmatrix}, \quad \dot{\rho} = \begin{pmatrix} & & E_{d-1}^* \\ p & & \\ & pE_{d-1}^* & \\ & & 1 \end{pmatrix}$$

of $\text{GO}_{2d}(\mathbb{Q}_p)$. They normalize T and satisfy

$$\begin{aligned} \dot{\omega}\dot{u} &= \dot{u}\dot{\omega}, & \dot{\omega}^2 &= p \cdot \text{id} \\ \dot{\omega}\dot{s}_i\dot{\omega}^{-1} &= \dot{s}_{d-i} & \text{for } 0 \leq i \leq d, \\ \dot{\rho}^2 &= p \cdot \dot{u}, \\ \dot{\rho}\dot{s}_i\dot{\rho}^{-1} &= \dot{s}_{d-i} & \text{for } 2 \leq i \leq d-2, \\ \dot{\rho}\dot{s}_{d-1}\dot{\rho}^{-1} &= \dot{s}_1, & \dot{\rho}\dot{s}_d\dot{\rho}^{-1} &= \dot{s}_0, & \dot{\rho}\dot{s}_0\dot{\rho}^{-1} &= \dot{s}_{d-1}, & \dot{\rho}\dot{s}_1\dot{\rho}^{-1} &= \dot{s}_d. \end{aligned}$$

The element $\dot{\omega}$ belongs to G if and only if d is even. The element $\dot{\rho}$ belongs to G if and only if d is odd.

Let $\mathcal{H}(G, I_0)'_{\text{aff},k}$ denote the k -sub algebra of $\mathcal{H}(G, I_0)_k$ generated by $\mathcal{H}(G, I_0)_{\text{aff},k}$ together with $T_{p \cdot \text{id}} = T_{\dot{\omega}^2}$ and $T_{p^{-1} \cdot \text{id}} = T_{p^{-1} \cdot \text{id}}$ if d is even, resp. $T_{p^2 \cdot \text{id}} = T_{\dot{\rho}^4}$ and $T_{p^{-2} \cdot \text{id}} = T_{p^{-2} \cdot \text{id}}$ if d is odd.

Suppose we are given a character $\lambda : \overline{T} \rightarrow k^\times$, a subset $\mathcal{J} \subset S_\lambda$ and some $b \in k^\times$. Define the numbers $0 \leq k_i = k_i(\lambda, \mathcal{J}) \leq p-1$ as in subsection 3.3. The character $\chi_{\lambda, \mathcal{J}}$ of $\mathcal{H}(G, I_0)_{\text{aff}, k}$ extends uniquely to a character

$$\chi_{\lambda, \mathcal{J}, b} : \mathcal{H}(G, I_0)'_{\text{aff}, k} \longrightarrow k$$

which sends $T_{p \cdot \text{id}}$ to b if d is even, resp. which sends $T_{p^2 \cdot \text{id}}$ to b if d is odd (see the proof of [8] Proposition 3). We define the $\mathcal{H}(G, I_0)_k$ -module

$$M = M[\lambda, \mathcal{J}, b] = \mathcal{H}(G, I_0)_k \otimes_{\mathcal{H}(G, I_0)'_{\text{aff}, k}} k.e$$

where $k.e$ denotes the one dimensional k -vector space on the basis element e , endowed with the action of $\mathcal{H}(G, I_0)'_{\text{aff}, k}$ by the character $\chi_{\lambda, \mathcal{J}, b}$. As a k -vector space, M has dimension 4. A k -basis is e_0, e_1, f_0, f_1 where we write

$$e_0 = 1 \otimes e, \quad f_0 = T_{\dot{u}} \otimes e, \quad e_1 = T_{\dot{\omega}} \otimes e, \quad f_1 = T_{\dot{u}\dot{\omega}} \otimes e \quad \text{if } d \text{ is even,}$$

$$e_0 = 1 \otimes e, \quad f_0 = T_{\dot{u}} \otimes e, \quad e_1 = T_{\dot{\rho}} \otimes e, \quad f_1 = T_{\dot{u}\dot{\rho}} \otimes e \quad \text{if } d \text{ is odd.}$$

Definition: We call an $\mathcal{H}(G, I_0)_k$ -module *standard supersingular* if it is isomorphic with $M[\lambda, \mathcal{J}, b]$ for some λ, \mathcal{J}, b such that $k_\bullet \notin \{(0, \dots, 0), (p-1, \dots, p-1)\}$.

For $2 \leq j \leq d-2$ let $\widetilde{j} = j$, and furthermore let $\widetilde{d-1} = d$ and $\widetilde{d} = d-1$ and $\widetilde{1} = 0$ and $\widetilde{0} = 1$. Letting $\widetilde{\beta} = (\cdot) \circ \beta$ we then have

$$\dot{u}\phi\dot{u}^{-1} = (p^n \cdot \text{id})\dot{s}_{\widetilde{\beta}(1)} \cdots \dot{s}_{\widetilde{\beta}(2d-2)}$$

with $n = 1$ if d is even, but $n = 2$ if d is odd. If d is odd we consider in addition the following two maps γ and δ from $[1, 2d-2]$ to $[0, d]$. We put $\gamma(1) = 1, \gamma(d-1) = d, \gamma(d) = 0$ and $\gamma(2d-2) = d-1$. We put $\delta(1) = 0, \delta(d-1) = d-1, \delta(d) = 1$ and $\delta(2d-2) = d$. We put $\gamma(i) = \delta(i) = \beta(2d-2-i)$ for all $i \in [2, \dots, d-2] \cup [d+1, \dots, 2d-3]$. We then have

$$\dot{\varrho}\phi\dot{\varrho}^{-1} = (p^2 \cdot \text{id})\dot{s}_{\gamma(1)} \cdots \dot{s}_{\gamma(2d-2)}, \quad \dot{\varrho}^{-1}\phi\dot{\varrho} = (p^2 \cdot \text{id})\dot{s}_{\delta(1)} \cdots \dot{s}_{\delta(2d-2)}.$$

Fix $M[\lambda, \mathcal{J}, b]$ for some λ, \mathcal{J}, b . Put

$$\begin{aligned} n_{e_0} &= \sum_{i=0}^{2d-3} k_{\beta(i+1)} p^i, & n_{f_0} &= \sum_{i=0}^{2d-3} k_{\widetilde{\beta}(i+1)} p^i & \text{for any parity of } d, \\ n_{e_1} &= \sum_{i=0}^{2d-3} k_{\beta(2d-2-i)} p^i, & n_{f_1} &= \sum_{i=0}^{2d-3} k_{\widetilde{\beta}(2d-2-i)} p^i & \text{if } d \text{ is even,} \end{aligned}$$

$$n_{e_1} = \sum_{i=0}^{2d-3} k_{\gamma(i+1)} p^i, \quad n_{f_1} = \sum_{i=0}^{2d-3} k_{\delta(i+1)} p^i \quad \text{if } d \text{ is odd.}$$

Let $0 \leq s_{e_0}, s_{f_0}, s_{e_1}, s_{f_1} \leq p-2$ be such that for all $x \in \mathbb{F}_p^\times$ we have

$$\begin{aligned} \lambda(\tau(x)) &= x^{-s_{e_0}}, & \lambda(\dot{u}\tau(x)\dot{u}^{-1}) &= x^{-s_{f_0}} & \text{for any parity of } d, \\ \lambda(\dot{\omega}\tau(x)\dot{\omega}^{-1}) &= x^{-s_{e_1}}, & \lambda(\dot{\omega}\dot{u}\tau(x)\dot{u}^{-1}\dot{\omega}^{-1}) &= x^{-s_{f_1}} & \text{if } d \text{ is even,} \\ \lambda(\dot{\rho}\tau(x)\dot{\rho}^{-1}) &= x^{-s_{e_1}}, & \lambda(\dot{\rho}\dot{u}\tau(x)\dot{u}^{-1}\dot{\rho}^{-1}) &= x^{-s_{f_1}} & \text{if } d \text{ is odd.} \end{aligned}$$

Put $\varrho = k_0!k_1!k_{d-1}!k_d! \prod_{i=2}^{d-2} (k_i!)^2 = \prod_{i=0}^{2d-3} (k_{\beta(i+1)}!) = \prod_{i=0}^{2d-3} (k_{\tilde{\beta}(i+1)}!)$.

Lemma 4.13. *The assignment $M[\lambda, \mathcal{J}, b] \mapsto (n_{e_0}, s_{e_0}, b\varrho^{-1})$ induces a bijection between the set of isomorphism classes of standard supersingular $\mathcal{H}(G, I_0)_k$ -modules and $\mathfrak{S}_D(2d-2)$.*

PROOF: We have $\alpha_0^\vee(x)\alpha_1^\vee(x)\alpha_{d-1}^\vee(x)\alpha_d^\vee(x) \prod_{i=2}^{d-2} (\alpha_i^\vee)^2(x) = 1$ for all $x \in \mathbb{F}_p^\times$ (as can be seen e.g. from formula (30)). This implies

$$(31) \quad k_0 + k_1 + k_{d-1} + k_d + 2 \sum_{i=2}^{d-2} k_i \equiv n_{e_0} \equiv n_{f_0} \equiv n_{e_1} \equiv n_{f_1} \equiv 0 \pmod{p-1}.$$

It follows from [8] Proposition 3 that $M[\lambda, \mathcal{J}, b]$ and $M[\lambda', \mathcal{J}', b']$ are isomorphic if and only if $b = b'$ and the pair (λ, \mathcal{J}) is conjugate with the pair (λ', \mathcal{J}') by means of $\dot{u}^n \dot{\omega}^m$ for some $n, m \in \{0, 1\}$ (if d is even), resp. by means of $\dot{u}^n \dot{\rho}^m$ for some $n, m \in \{0, 1\}$ (if d is odd). Under the map $M[\lambda, \mathcal{J}, b] \mapsto (n_{e_0}, s_{e_0}, b\varrho^{-1})$, conjugation by \dot{u} corresponds to the permutation ι_0 of $\tilde{\mathfrak{S}}_D(2d-2)$, while conjugation by $\dot{\omega}$, resp. by $\dot{\rho}$, corresponds to the permutation ι_1 of $\tilde{\mathfrak{S}}_D(2d-2)$. We may thus proceed as in the proof of Lemma 4.2 to see that our mapping is well defined and bijective. \square

Lemma 4.14. *For an appropriate choice of the isomorphism Θ we have, in $H_0(\overline{\mathfrak{X}}_+, \Theta_* \mathcal{V}_M)$,*

$$\begin{aligned} t^{n_{e_j}} \varphi^{2d-2} e_j &= \varrho b^{-1} e_j, \\ t^{n_{f_j}} \varphi^{2d-2} f_j &= \varrho b^{-1} f_j, \\ \gamma(x) e_j &= x^{-s_{e_j}} e_j, \\ \gamma(x) f_j &= x^{-s_{f_j}} f_j \end{aligned}$$

for $x \in \mathbb{F}_p^\times$ and $j = 0, 1$. The action of Γ_0 on $H_0(\overline{\mathfrak{X}}_+, \Theta_* \mathcal{V}_M)$ is trivial on the subspace M .

PROOF: As in Lemma 4.4. \square

Corollary 4.15. *The étale (φ^{2d-2}, Γ) -module over $k_{\mathcal{E}}$ associated with $H_0(\overline{\mathfrak{X}}_+, \Theta_* \mathcal{V}_M)$ admits a $k_{\mathcal{E}}$ -basis $g_{e_0}, g_{f_0}, g_{e_1}, g_{f_1}$ such that for both $j = 0$ and $j = 1$ we have*

$$\begin{aligned}\varphi^{2d-2} g_{e_j} &= b \varrho^{-1} t^{n_{e_j}+1-p^{2d-2}} g_{e_j} \\ \varphi^{2d-2} g_{f_j} &= b \varrho^{-1} t^{n_{f_j}+1-p^{2d-2}} g_{f_j} \\ \gamma(x)(g_{e_j}) - x^{s_{e_j}} g_{e_j} &\in t \cdot k_{\mathcal{E}}^+ \cdot g_{e_j} \\ \gamma(x)(g_{f_j}) - x^{s_{f_j}} g_{f_j} &\in t \cdot k_{\mathcal{E}}^+ \cdot g_{f_j}\end{aligned}$$

PROOF: This follows from Lemma 4.14 as explained in [2] Lemma 6.4. \square

Corollary 4.16. *The functor $M \mapsto \mathbf{D}(\Theta_* \mathcal{V}_M)$ induces a bijection between*

- (a) *the set of isomorphism classes of standard supersingular $\mathcal{H}(G, I_0)_k$ -modules, and*
- (b) *the set of isomorphism classes of D -symmetric étale (φ^{2d-2}, Γ) -modules over $k_{\mathcal{E}}$.*

PROOF: We first verify that $\mathbf{D}(\Theta_* \mathcal{V}_M)$ for $M = M[\lambda, \mathcal{J}, b]$ is indeed D -symmetric. For this let $\mathbf{D}_{11} = \langle g_{e_0} \rangle$, $\mathbf{D}_{12} = \langle g_{f_0} \rangle$, $\mathbf{D}_{21} = \langle g_{e_1} \rangle$, $\mathbf{D}_{22} = \langle g_{f_1} \rangle$. Then $k_i(\mathbf{D}_{11}) = k_{\beta(i+1)}$ and $k_i(\mathbf{D}_{12}) = k_{\tilde{\beta}(i+1)}$; moreover $k_i(\mathbf{D}_{21}) = k_{\beta(2d-d-i)}$ and $k_i(\mathbf{D}_{22}) = k_{\tilde{\beta}(2d-d-i)}$ if d is even, but $k_i(\mathbf{D}_{21}) = k_{\gamma(i+1)}$ and $k_i(\mathbf{D}_{22}) = k_{\delta(i+1)}$ if d is odd.

For the condition on $s_{f_0} - s_{e_0} = s(\mathbf{D}_{12}) - s(\mathbf{D}_{11})$ we compute

$$\tau(x) \cdot \dot{u} \tau^{-1}(x) \dot{u}^{-1} = \text{diag}(x, E_{d-2}, x^{-1}, x^{-1}, E_{d-2}, x) = \left(\sum_{i=1}^{d-1} \alpha_i^{\vee} \right)(x),$$

hence application of λ gives $x^{s_{f_0} - s_{e_0}} = x^{\sum_{i=1}^{d-1} k_i}$ and hence $s_{f_0} - s_{e_0} \equiv \sum_{i=1}^{d-1} k_i = \sum_{i=0}^{d-2} k_i(\mathbf{D}_{11})$ modulo $(p-1)$. The condition on $s_{f_1} - s_{e_1} = s(\mathbf{D}_{22}) - s(\mathbf{D}_{21})$ in case d is even is exactly verified like the one for $s_{f_0} - s_{e_0}$ because $\dot{\omega} \tau(x) \dot{\omega}^{-1} \cdot \dot{\omega} \dot{u} \tau^{-1}(x) \dot{u}^{-1} \dot{\omega}^{-1} = \tau(x) \cdot \dot{u} \tau^{-1}(x) \dot{u}^{-1}$. In case d is odd the computation is

$$\dot{\rho} \tau(x) \dot{\rho}^{-1} \cdot \dot{\rho} \dot{u} \tau^{-1}(x) \dot{u}^{-1} \dot{\rho}^{-1} = \text{diag}(x, E_{d-2}, x, x^{-1}, E_{d-2}, x^{-1}) = (\alpha_d^{\vee} + \sum_{i=1}^{d-2} \alpha_i^{\vee})(x),$$

hence $s_{f_1} - s_{e_1} \equiv k_d + \sum_{i=1}^{d-2} k_i = s(\mathbf{D}_{12}) - s(\mathbf{D}_{11}) + k_d - k_{d-1} = \sum_{i=0}^{d-2} k_i(\mathbf{D}_{21})$ modulo $(p-1)$.

To see the condition on $s_{e_1} - s_{e_0} = s(\mathbf{D}_{21}) - s(\mathbf{D}_{11})$ in case d is even we compute

$$\begin{aligned}(32) \quad \tau(x) \cdot \dot{\omega} \tau^{-1}(x) \dot{\omega}^{-1} &= \text{diag}(1, x E_{d-2}, 1, 1, x^{-1} E_{d-2}, 1) \\ &= \left(\frac{d-2}{2} \alpha_{d-1}^{\vee} + \frac{d-2}{2} \alpha_d^{\vee} + \sum_{i=2}^{d-2} (i-1) \alpha_i^{\vee} \right)(x),\end{aligned}$$

hence application of λ gives $x^{s_{e_1}-s_{e_0}} = x^{\frac{d-2}{2}k_{d-1}+\frac{d-2}{2}k_d+\sum_{i=2}^{d-2}(i-1)k_i}$ and hence $s_{e_1} - s_{e_0} \equiv \frac{d-2}{2}k_{d-1} + \frac{d-2}{2}k_d + \sum_{i=2}^{d-2}(i-1)k_i = \frac{d-2}{2}(k_{d-1}(\mathbf{D}_{11}) + k_0(\mathbf{D}_{11})) + \sum_{i=2}^{d-2}(i-1)k_{d-i-1}(\mathbf{D}_{11})$ modulo $(p-1)$. If however d is odd we compute

$$\begin{aligned}\tau(x) \cdot \dot{\rho}\tau^{-1}(x)\dot{\rho}^{-1} &= \text{diag}(1, xE_{d-2}, x^{-1}, 1, x^{-1}E_{d-2}, x) \\ &= \left(\frac{d-1}{2}\alpha_{d-1}^\vee + \frac{d-3}{2}\alpha_d^\vee + \sum_{i=2}^{d-2}(i-1)\alpha_i^\vee\right)(x),\end{aligned}$$

hence application of λ gives $x^{s_{e_1}-s_{e_0}} = x^{\frac{d-1}{2}k_{d-1}+\frac{d-3}{2}k_d+\sum_{i=2}^{d-2}(i-1)k_i}$ and hence $s_{e_1} - s_{e_0} \equiv \frac{d-1}{2}k_{d-1} + \frac{d-3}{2}k_d + \sum_{i=2}^{d-2}(i-1)k_i = \frac{d-3}{2}k_{\frac{r}{2}}(\mathbf{D}_{11}) + \frac{d-1}{2}k_0(\mathbf{D}_{11}) + \sum_{i=2}^{d-2}(i-1)k_{d-i-1}(\mathbf{D}_{11})$ modulo $(p-1)$. We have shown that $\mathbf{D}(\Theta_*\mathcal{V}_M)$ is D -symmetric.

One checks that composing $M \mapsto \mathbf{D}(\Theta_*\mathcal{V}_M)$ with the map of Lemma 2.6 gives the map of Lemma 4.13. As the maps in Lemmata 4.13 and 2.6 are bijective we obtain our result. \square

Remark: Consider the subgroup $G' = \text{SO}_{2d}(\mathbb{Q}_p)$ of G . If we replace the above τ by $\tau : \mathbb{Z}_p^\times \rightarrow T_0$, $x \mapsto \text{diag}(xE_{d-1}, x^{-1}E_d, x)$, and if we replace the above ϕ by $\phi = \dot{s}_{d-1}\dot{s}_{d-2} \cdots \dot{s}_2\dot{s}_1\dot{s}_d\dot{s}_{d-2}\dot{s}_{d-3} \cdots \dot{s}_3\dot{s}_2\dot{s}_0$, then everything in fact happens inside G' , and there is no dichotomy between d even or odd. We then have $\alpha^{(j)} \circ \tau = \text{id}_{\mathbb{Z}_p^\times}^2$ for all $j \geq 0$. Let $\text{Mod}_0^{\text{fin}}\mathcal{H}(G', G' \cap I_0)$ denote the category of finite- \mathfrak{o} -length $\mathcal{H}(G', G' \cap I_0)$ -modules on which $\tau(-1)$ (i.e. $T_{\tau(-1)} = T_{\tau(-1)^{-1}}$) acts trivially. For $M \in \text{Mod}_0^{\text{fin}}\mathcal{H}(G', G' \cap I_0)$ we obtain an action of $[\mathfrak{N}_0, \varphi^{2d-2}, \Gamma^2]$ on $H_0(\tilde{\mathcal{X}}_+, \Theta_*\mathcal{V}_M)$, where $\Gamma^2 = \{\gamma^2 \mid \gamma \in \Gamma\} \subset \Gamma$. Correspondingly, we obtain a functor from $\text{Mod}_0^{\text{fin}}\mathcal{H}(G', G' \cap I_0)$ to the category of $(\varphi^{2d-2}, \Gamma^2)$ -modules over $\mathcal{O}_{\mathcal{E}}$.

Remark: Instead of the element $\phi \in N(T)$ used above we might also work with the element $\dot{s}_{d-1} \cdots \dot{s}_2\dot{s}_1\dot{u}$ of length $d-1$ (or products of this with elements of $p^{\mathbb{Z}} \cdot \text{id}$), keeping the same $C^{(\bullet)}$. This results in a functor from $\mathcal{H}(G, I_0)$ -modules to (φ^{d-1}, Γ) -modules. Up to a factor in $p^{\mathbb{Z}} \cdot \text{id}$, the square of $\dot{s}_{d-1} \cdots \dot{s}_2\dot{s}_1\dot{u}$ is the element ϕ used above.

Remark: For the affine root system of type D_d there are three co minuscule fundamental coweights (cf. [1] chapter 8, par 7.3). The corresponding ϕ 's for the other two choices are longer.

4.4 Affine root system \tilde{A}_d

Assume $d \geq 1$ and consider $G = \text{GL}_{d+1}(\mathbb{Q}_p)$. Let

$$\dot{u} = \begin{pmatrix} & E_d \\ p & \end{pmatrix}.$$

For $1 \leq i \leq d$ let

$$\dot{s}_i = \text{diag}(E_{i-1}, S_1, E_{d-i})$$

and let $\dot{s}_0 = \dot{u}\dot{s}_1\dot{u}^{-1}$. Let T be the maximal torus consisting of diagonal matrices. Let Φ^+ be such that $N = \prod_{\alpha \in \Phi^+} N_\alpha$ is the subgroup of upper triangular unipotent matrices. Let I_0 be the subgroup consisting of elements in $\text{GL}_{d+1}(\mathbb{Z}_p)$ which are upper triangular modulo p . We put

$$\phi = (p \cdot \text{id})\dot{s}_d \cdots \dot{s}_0 = (p \cdot \text{id})\dot{s}_{\beta(1)} \cdots \dot{s}_{\beta(d+1)}$$

where $\beta(i) = d+1-i$ for $1 \leq i \leq d+1$. For $a, b \in \mathbb{Z}_{\geq 0}$ with $0 \leq b < d+1$ we put

$$C^{(a(d+1)+b)} = \phi^a s_d \cdots s_{d-b+1} C = \phi^a s_{\beta(1)} \cdots s_{\beta(b)} C.$$

We define the homomorphism

$$\tau : \mathbb{Z}_p^\times \longrightarrow T_0, \quad x \mapsto \text{diag}(E_d, x^{-1}).$$

The sequence $C = C^{(0)}, C^{(1)}, C^{(2)}, \dots$ satisfies hypothesis (4). The corresponding $\alpha^{(j)} \in \Phi^+$ for $j \geq 0$ satisfy $\alpha^{(j)} \circ \tau = \text{id}_{\mathbb{Z}_p^\times}$, and we have $\tau(a)\phi = \phi\tau(a)$ for all $a \in \mathbb{Z}_p^\times$. We thus obtain a functor from $\text{Mod}^{\text{fin}}(\mathcal{H}(G, I_0))$ to the category of (φ^{d+1}, Γ) -modules over \mathcal{O}_E .

Let $\mathcal{H}(G, I_0)'_{\text{aff}, k}$ denote the k -sub algebra of $\mathcal{H}(G, I_0)_k$ generated by $\mathcal{H}(G, I_0)_{\text{aff}, k}$ together with $T_{p \cdot \text{id}} = T_{\dot{u}^{d+1}}$ and $T_{p \cdot \text{id}}^{-1} = T_{p^{-1} \cdot \text{id}}$.

Suppose we are given a character $\lambda : \overline{T} \rightarrow k^\times$, a subset $\mathcal{J} \subset S_\lambda$ and some $b \in k^\times$. Define the numbers $0 \leq k_i = k_i(\lambda, \mathcal{J}) \leq p-1$ as in subsection 3.3. The character $\chi_{\lambda, \mathcal{J}}$ of $\mathcal{H}(G, I_0)_{\text{aff}, k}$ extends uniquely to a character

$$\chi_{\lambda, \mathcal{J}, b} : \mathcal{H}(G, I_0)'_{\text{aff}, k} \longrightarrow k$$

which sends $T_{p \cdot \text{id}}$ to b (see the proof of [8] Proposition 3). Define the $\mathcal{H}(G, I_0)_k$ -module

$$M = M[\lambda, \mathcal{J}, b] = \mathcal{H}(G, I_0)_k \otimes_{\mathcal{H}(G, I_0)'_{\text{aff}, k}} k.e$$

where $k.e$ denotes the one dimensional k -vector space on the basis element e , endowed with the action of $\mathcal{H}(G, I_0)'_{\text{aff}, k}$ by the character $\chi_{\lambda, \mathcal{J}, b}$. As a k -vector space, M has dimension $d+1$, a k -basis is $\{e_i\}_{0 \leq i \leq d}$ where we write $e_i = T_{\dot{u}^{-i}} \otimes e$.

Definition: We call an $\mathcal{H}(G, I_0)_k$ -module *standard supersingular* if it is isomorphic with $M[\lambda, \mathcal{J}, b]$ for some λ, \mathcal{J}, b such that $k_\bullet \notin \{(0, \dots, 0), (p-1, \dots, p-1)\}$.

For $0 \leq j \leq d$ put $n_{e_j} = \sum_{i=0}^d k_{j-i} p^i$ (reading $j-i$ as its representative modulo $(d+1)$ in $[0, d]$) and let s_{e_j} be such that $\lambda(\dot{u}^{-j} \tau(x) \dot{u}^j) = x^{-s_{e_j}}$. Put $\varrho = \lambda(-\text{id}) \prod_{i=0}^d (k_i!)$.

Theorem 4.17. *The étale (φ^{d+1}, Γ) -module $\mathbf{D}(\Theta_* \mathcal{V}_M)$ over $k_{\mathcal{E}}$ associated with $H_0(\overline{\mathfrak{X}}_+, \Theta_* \mathcal{V}_M)$ admits a $k_{\mathcal{E}}$ -basis $\{g_{e_j}\}_{0 \leq j \leq d}$ such that for all j we have*

$$\begin{aligned}\varphi^{d+1} g_{e_j} &= b \varrho^{-1} t^{n_{e_j}+1-p^{d+1}} g_{e_j}, \\ \gamma(x) g_{e_j} - x^{s_{e_j}} g_{e_j} &\in t \cdot k_{\mathcal{E}}^+ \cdot g_{e_j}.\end{aligned}$$

The functor $M \mapsto \mathbf{D}(\Theta_ \mathcal{V}_M)$ induces a bijection between*

- (a) *the set of isomorphism classes of standard supersingular $\mathcal{H}(G, I_0)_k$ -modules, and*
- (b) *the set of isomorphism classes of A -symmetric étale (φ^{d+1}, Γ) -modules over $k_{\mathcal{E}}$.*

PROOF: For the formulae describing $\mathbf{D}(\Theta_* \mathcal{V}_M)$ one may proceed exactly as in the proof of Corollary 4.5. (The only tiny additional point to be observed is that the \dot{s}_i (in keeping with our choice in [2]) do not lie in the images of the ι_{α_i} ; this is accounted for by the sign factor $\lambda(-\text{id})$ in the definition of ϱ .) Alternatively, as our ϕ is the $(d+1)$ -st power of the ϕ considered in section 8 of [2], the computations of loc. cit. may be carried over.

To see that $\mathbf{D}(\Theta_* \mathcal{V}_M)$ is A -symmetric put $\mathbf{D}_j = \langle g_{e_j} \rangle$ for $0 \leq j \leq d$ and compare the above formulae with those defining A -symmetry; e.g. we find $s_{e_0} - s_{e_j} \equiv \sum_{i=1}^j k_i$ modulo $(p-1)$. The bijectivity statement is then verified as before. \square

Remark: Application of the functor of Lemma 2.1 to any one of the direct summands \mathbf{D}_j of the A -symmetric étale (φ^{d+1}, Γ) -module $\mathbf{D}(\Theta_* \mathcal{V}_M)$ yields an étale (φ, Γ) -module isomorphic with the one assigned to M in [2].

Remark: Consider the subgroup $G' = \text{SL}_{d+1}(\mathbb{Q}_p)$ of G . If we replace the above τ by $\tau : \mathbb{Z}_p^\times \rightarrow T_0$, $x \mapsto \text{diag}(xE_d, x^{-d})$ and if we replace the above ϕ by $\phi = \dot{s}_d \dot{s}_{d-1} \cdots \dot{s}_1 \dot{s}_0$ then everything in fact happens inside G' . We then have $\alpha^{(j)} \circ \tau = \text{id}_{\mathbb{Z}_p^\times}^{d+1}$ for all $j \geq 0$. Let $\text{Mod}_0^{\text{fin}} \mathcal{H}(G', G' \cap I_0)$ denote the category of finite- \mathfrak{o} -length $\mathcal{H}(G', G' \cap I_0)$ -modules on which the xE_{d+1} (i.e. the $T_{x^{-1}E_{d+1}}$) for all $x \in \mathbb{Z}_p^\times$ with $x^{d+1} = 1$ act trivially. (Notice that $\tau(x) = xE_{d+1}$ for such x .) For $M \in \text{Mod}_0^{\text{fin}} \mathcal{H}(G', G' \cap I_0)$ we obtain an action of $[\mathfrak{N}_0, \varphi^{d+1}, \Gamma^{d+1}]$ on $H_0(\overline{\mathfrak{X}}_+, \Theta_* \mathcal{V}_M)$, where $\Gamma^{d+1} = \{\gamma^{d+1} \mid \gamma \in \Gamma\} \subset \Gamma$. Correspondingly, we obtain a functor from $\text{Mod}_0^{\text{fin}} \mathcal{H}(G', G' \cap I_0)$ to the category of $(\varphi^{d+1}, \Gamma^{d+1})$ -modules over $\mathcal{O}_{\mathcal{E}}$.

Remark: As all the fundamental coweights τ of T are minuscule, each of them admits ϕ 's for which the pair (ϕ, τ) satisfies the properties asked for in Lemma 3.1. For example, let $1 \leq g \leq d$. For $\phi = \dot{s}_g \cdot \dot{s}_{g+1} \cdots \dot{s}_d \cdot \dot{u}$ as well as for $\phi = \dot{s}_g \cdot \dot{s}_{g-1} \cdots \dot{s}_1 \cdot \dot{u}^{-1}$ there is a unique minimal gallery from C to $\phi(C)$ which admits a ϕ -periodic continuation to a gallery (3),

giving rise to a functor from $\text{Mod}^{\text{fn}}(\mathcal{H}(G, I_0))$ to the category of (φ^r, Γ) -modules over $\mathcal{O}_{\mathcal{E}}$, where $r = \ell(\phi)$.

5 Exceptional groups of type \tilde{E}_6 and \tilde{E}_7

Let G be the group of \mathbb{Q}_p -rational points of a \mathbb{Q}_p -split connected reductive group over \mathbb{Q}_p with connected center Z . Fix a maximal \mathbb{Q}_p -split torus T and define Φ , $N(T)$, W , \widehat{W} and W_{aff} as before.

5.1 Affine root system \tilde{E}_6

Assume that the root system Φ is of type E_6 . Following [1] (for the indexing) we then have generators s_1, \dots, s_6 of W and s_0, \dots, s_6 of W_{aff} such that

$$(s_1 s_3)^3 = (s_3 s_4)^3 = (s_4 s_5)^3 = (s_5 s_6)^3 = (s_2 s_4)^3 = (s_0 s_2)^3 = 1$$

and moreover $(s_i s_j)^2 = 1$ for all other pairs $i < j$, and $s_i^2 = 1$ for all i . In the extended affine Weyl group \widehat{W} we find (cf. [4]) an element u of length 0 with

$$(33) \quad \begin{aligned} u^3 &= 1 & \text{and} & & u s_4 u^{-1} &= s_4, \\ u s_3 u^{-1} &= s_5, & u s_5 u^{-1} &= s_2, & u s_2 u^{-1} &= s_3, \\ u s_1 u^{-1} &= s_6, & u s_6 u^{-1} &= s_0, & u s_0 u^{-1} &= s_1. \end{aligned}$$

(We have $\widehat{W} = W_{\text{aff}} \rtimes W_{\Omega}$ with the three-element subgroup $W_{\Omega} = \{1, u, u^2\}$.) Let e_1, \dots, e_8 denote the standard basis of \mathbb{R}^8 . We use the standard inner product $\langle \cdot, \cdot \rangle$ on \mathbb{R}^8 to view both the root system Φ as well as its dual Φ^{\vee} as living inside \mathbb{R}^8 . We choose a positive system Φ^+ in Φ such that, as in [1], the simple roots are $\alpha_1 = \alpha_1^{\vee} = \frac{1}{2}(e_1 + e_8 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7)$, $\alpha_2 = \alpha_2^{\vee} = e_2 + e_1$, $\alpha_3 = \alpha_3^{\vee} = e_2 - e_1$, $\alpha_4 = \alpha_4^{\vee} = e_3 - e_2$, $\alpha_5 = \alpha_5^{\vee} = e_4 - e_3$, $\alpha_6 = \alpha_6^{\vee} = e_5 - e_4$ while the negative of the highest root is $\alpha_0 = \alpha_0^{\vee} = \frac{1}{2}(e_6 + e_7 - e_1 - e_2 - e_3 - e_4 - e_5 - e_8)$. The set of positive roots is

$$\Phi^+ = \{e_j \pm e_i \mid 1 \leq i < j \leq 5\} \cup \left\{ \frac{1}{2}(-e_6 - e_7 + e_8 + \sum_{i=1}^5 (-1)^{\nu_i} e_i) \mid \sum_{i=1}^5 \nu_i \text{ even} \right\}.$$

We lift u and the s_i to elements \dot{u} and \dot{s}_i in $N(T)$. We then put

$$(34) \quad \phi = \dot{s}_2 \dot{s}_4 \dot{s}_3 \dot{s}_1 \dot{u}^{-1} \in N(T).$$

We define ∇ as in section 3.1.

Proposition 5.1. *There is a $\tau \in \nabla$ such that the pair (ϕ, τ) satisfies the hypotheses of Lemma 3.1. More precisely, ϕ is power multiplicative, and for the minuscule fundamental (co)weight $\tau = \omega_1 = \frac{2}{3}(e_8 - e_7 - e_6) \in \nabla$ we have $\phi^{12} = \tau^3$ in W_{aff} .*

PROOF: (Here τ^3 designates the three fold iterate of translation by $\tau = \omega_1$, i.e. translation by the element $3\omega_1$ of the root lattice.) We define the subset

$$\Phi(\omega_1) = \left\{ \frac{1}{2}(-e_6 - e_7 + e_8 + \sum_{i=1}^5 (-1)^{\nu_i} e_i) \mid \sum_{i=1}^5 \nu_i \text{ even} \right\}$$

of Φ^+ . We compute $\langle \beta, 3\omega_1 \rangle = 3$ for each $\beta \in \Phi(\omega_1)$, but $\langle \beta, 3\omega_1 \rangle = 0$ for each $\beta \in \Phi^+ - \Phi(\omega_1)$. This means that for each $\beta \in \Phi(\omega_1)$ the translation by $3\omega_1$ crosses 3 walls parallel to the hyperplane corresponding to β , and that for each $\beta \in \Phi^+ - \Phi(\omega_1)$ it crosses no wall parallel to the hyperplane corresponding to β . As $\Phi(\omega_1)$ contains 16 elements it follows that $\ell(\tau^3) = \ell(3\omega_1) = 3 \cdot 16 = 48$. On the other hand, in the appendix we explain how one can prove $\phi^{12} = 3\omega_1$ (as elements in W_{aff}). Combining these two facts we deduce $\ell(\phi^{12}) = 48$ and hence that ϕ (which a priori has length ≤ 4) is power multiplicative. \square

Remark: Instead of verifying $\phi^{12} = 3\omega_1$ by means of a direct (computer) computation, one might try to prove that the affine transformation ϕ^{12} on A crosses exactly the same walls as does $3\omega_1$ (which are described in the above proof).

As explained in subsection 3.2 we now obtain a functor from $\text{Mod}^{\text{fin}}(\mathcal{H}(G, I_0))$ to the category of (φ^4, Γ) -modules over $\mathcal{O}_{\mathcal{E}}$.

Similarly, we may replace ϕ by its third power ϕ^3 which (in contrast to ϕ) is an element of W_{aff} (modulo T_0). It yields a functor from $\text{Mod}^{\text{fin}}(\mathcal{H}(G, I_0))$ to the category of (φ^{12}, Γ) -modules over $\mathcal{O}_{\mathcal{E}}$. As in our treatment of the cases C , B , D and A , this functor identifies the set of standard supersingular $\mathcal{H}(G, I_0)_k$ -modules bijectively with a set of certain E -symmetric étale (φ^{12}, Γ) -modules over $k_{\mathcal{E}}$ of dimension 3. We leave the details to the reader.

Remarks: (a) Dual to the above choice of ϕ is the choice

$$(35) \quad \phi = \dot{s}_2 \dot{s}_4 \dot{s}_5 \dot{s}_6 \dot{u} \in N(T).$$

For this choice, Proposition 5.1 holds true verbatim the same way, but now with the minuscule fundamental (co)weight $\tau = \omega_6 = \frac{1}{3}(3e_5 + e_8 - e_7 - e_6)$ with its corresponding subset (again containing 16 elements)

$$\Phi(\omega_6) = \left\{ \frac{1}{2}(e_5 - e_6 - e_7 + e_8 + \sum_{i=1}^4 (-1)^{\nu_i} e_i) \mid \sum_{i=1}^4 \nu_i \text{ even} \right\} \cup \{e_5 \pm e_i \mid 1 \leq i < 5\}$$

of Φ^+ . Again see the appendix.

(b) For ϕ given by either (34) or (35), consider the corresponding reduced expression of ϕ^3 in W_{aff} (obtained by three fold concatenation and then conjugation of the s_i 's appearing with powers of u so that no factor u or u^2 remains). The number of occurrences of the s_i are precisely the coefficients of the α_i^\vee in

$$\alpha_0^\vee + \alpha_1^\vee + \alpha_6^\vee + 2\alpha_2^\vee + 2\alpha_3^\vee + 2\alpha_5^\vee + 3\alpha_4^\vee = 0.$$

5.2 Affine root system \tilde{E}_7

Assume that the root system Φ is of type E_7 . Following [1] we then have generators s_1, \dots, s_7 of W and s_0, \dots, s_7 of W_{aff} such that

$$(s_0 s_1)^3 = (s_1 s_3)^3 = (s_3 s_4)^3 = (s_4 s_5)^3 = (s_5 s_6)^3 = (s_6 s_7)^3 = (s_2 s_4)^3 = 1$$

and moreover $(s_i s_j)^2 = 1$ for all other pairs $i < j$, and $s_i^2 = 1$ for all i . In the extended affine Weyl group \widehat{W} we find (cf. [4]) an element u of length 0 with

$$\begin{aligned} u^2 &= 1 & \text{and} & & u s_4 u &= s_4, & u s_2 u &= s_2, \\ u s_3 u &= s_5, & u s_6 u &= s_1, & u s_7 u &= s_0, \\ u s_5 u &= s_3, & u s_1 u &= s_6, & u s_0 u &= s_7. \end{aligned}$$

(We have $\widehat{W} = W_{\text{aff}} \rtimes W_\Omega$ with the two-element subgroup $W_\Omega = \{1, u\}$.) Let e_1, \dots, e_8 denote the standard basis of \mathbb{R}^8 . We use the standard inner product $\langle \cdot, \cdot \rangle$ on \mathbb{R}^8 to view both the root system Φ as well as its dual Φ^\vee as living inside \mathbb{R}^8 . We choose a positive system Φ^+ in Φ such that, as in [1], the simple roots are $\alpha_1 = \alpha_1^\vee = \frac{1}{2}(e_1 + e_8 - e_2 - e_3 - e_4 - e_5 - e_6 - e_7)$, $\alpha_2 = \alpha_2^\vee = e_2 + e_1$, $\alpha_3 = \alpha_3^\vee = e_2 - e_1$, $\alpha_4 = \alpha_4^\vee = e_3 - e_2$, $\alpha_5 = \alpha_5^\vee = e_4 - e_3$, $\alpha_6 = \alpha_6^\vee = e_5 - e_4$, $\alpha_7 = \alpha_7^\vee = e_6 - e_5$ while the negative of the highest root is $\alpha_0 = \alpha_0^\vee = e_7 - e_8$. The set of positive roots is

$$\Phi^+ = \{e_j \pm e_i \mid 1 \leq i < j \leq 6\} \cup \{e_8 - e_7\} \cup \left\{ \frac{1}{2}(e_8 - e_7 + \sum_{i=1}^6 (-1)^{\nu_i} e_i) \mid \sum_{i=1}^6 \nu_i \text{ odd} \right\}.$$

We lift u and the s_i to elements \dot{u} and \dot{s}_i in $N(T)$. We then put

$$\phi = \dot{s}_1 \dot{s}_3 \dot{s}_4 \dot{s}_2 \dot{s}_5 \dot{s}_4 \dot{s}_3 \dot{s}_1 \dot{s}_0 \dot{u} \in N(T).$$

We define ∇ as in section 3.1.

Proposition 5.2. *There is a $\tau \in \nabla$ such that the pair (ϕ, τ) satisfies the hypotheses of Lemma 3.1. More precisely, ϕ is power multiplicative, and for the minuscule fundamental (co)weight $\tau = \omega_7 = e_6 + \frac{1}{2}(e_8 - e_7) \in \nabla$ we have $\phi^6 = \tau^2$ in W_{aff} .*

PROOF: Exactly the same as for Propostion 5.1. The corresponding subset in Φ^+ is

$$\left\{\frac{1}{2}(e_6 + e_8 - e_7 + \sum_{i=1}^5 (-1)^{\nu_i} e_i) \mid \sum_{i=1}^5 \nu_i \text{ odd}\right\} \cup \{e_8 - e_7\} \cup \{e_6 \pm e_i \mid 1 \leq i < 6\}.$$

It contains 27 elements, thus $\ell(2\omega_7) = 54$. For a computer proof of $\phi^6 = 2\omega_7$ see the appendix. \square

As explained in subsection 3.2 we now obtain a functor from $\text{Mod}^{\text{fn}}(\mathcal{H}(G, I_0))$ to the category of (φ^9, Γ) -modules over $\mathcal{O}_{\mathcal{E}}$.

Similarly, we may replace ϕ by its square ϕ^2 which (in contrast to ϕ) is an element of W_{aff} (modulo T_0). It yields a functor from $\text{Mod}^{\text{fn}}(\mathcal{H}(G, I_0))$ to the category of (φ^{18}, Γ) -modules over $\mathcal{O}_{\mathcal{E}}$. Again this functor identifies the set of standard supersingular $\mathcal{H}(G, I_0)_k$ -modules bijectively with a set of certain E -symmetric étale (φ^{18}, Γ) -modules over $k_{\mathcal{E}}$ of dimension 2. We leave the details to the reader.

Remark: The number of occurencess of the s_i in $\phi^2 \in W_{\text{aff}}$ (cf. the case \tilde{E}_6) are the coefficients of the α_i^{\vee} in

$$\alpha_0^{\vee} + \alpha_7^{\vee} + 2(\alpha_1^{\vee} + \alpha_2^{\vee} + \alpha_6^{\vee}) + 3(\alpha_3^{\vee} + \alpha_5^{\vee}) + 4\alpha_4^{\vee} = 0.$$

6 Appendix

Verification of the statement $\phi^{12} = 3\omega_1$ in the proof of Proposition 5.1.

In the computer algebra system *sage*, the input

```
R=RootSystem(["E",6,1]).weight_lattice()
Lambda=R.fundamental_weights()
omega1=Lambda[1]-Lambda[0]
R.reduced_word_of_translation(3*omega1)
```

prompts the output

$$(36) \quad [0, 2, 4, 3, 5, 4, 2, 0, 6, 5, 4, 2, 3, 1, 4, 3, 5, 4, 2, 0, 6, 5, 4, 2, 3, 1, 4, 3, 5, 4, 2, 0, 6, 5, 4, 2, 3, 1, 4, 3, 5, 4, 2, 6, 5, 4, 3, 1].$$

By definition of the function *reduced_word_of_translation* this means $s_{i_1}^* \cdots s_{i_{48}}^* = 3\omega_1$, with the string $[i_1, \dots, i_{48}]$ as given by (36). Here $s_i^* = s_i$ for $1 \leq i \leq 6$, but s_0^* denotes the affine reflection in the outer face of Bourbaki's fundamental alcove A . Since we deviate from these conventions in that our s_0 is the affine reflection in the outer face of the *negative* $C = -A$ of A , we must modify the above string (36) as follows. First, writing $s_i^{**} = s_0^* s_i^* s_0^*$

for $0 \leq i \leq 6$, conjugating the factors in the previous word by s_0^* and commuting some of its factors where allowed, the above says $s_{j_1}^{**} \cdots s_{j_{48}}^{**} = 3\omega_1$ where the string $[j_1, \dots, j_{48}]$ is given by

$$(37) \quad \begin{aligned} & [2, 4, 5, 6, 3, 4, 2, 0, 5, 4, 3, 1, 2, 4, 5, 6, 3, 4, 2, 0, 5, 4, 3, 1, \\ & 2, 4, 5, 6, 3, 4, 2, 0, 5, 4, 3, 1, 2, 4, 5, 6, 3, 4, 2, 0, 5, 4, 3, 1]. \end{aligned}$$

The s_i^{**} are precisely the reflections in the codimension 1 faces of s_0^*A . But s_0^*A is a translate of C , and under this translation, the reflection $s_0^{**} = s_0^*$ corresponds to s_0 , whereas for $1 \leq i \leq 6$ the reflection s_i^{**} corresponds to $w_0 s_i w_0$, where w_0 is the longest element of W . We have $w_0 s_i w_0 = s_i$ for $i \in \{0, 2, 4\}$, but $w_0 s_3 w_0 = s_5$ and $w_0 s_1 w_0 = s_6$. Thus, we obtain $s_{k_1} \cdots s_{k_{48}} = 3\omega_1$ where the string $[k_1, \dots, k_{48}]$ is obtained from the string (37) by keeping its entry values 0, 2 and 4, while exchanging the entry values 3 with 5 and 1 with 6. Using formulae (33) one checks that $s_{k_1} \cdots s_{k_{48}} = \phi^{12}$. \square

Verification of the statement $\phi^{12} = 3\omega_6$ for ϕ given by (35).

The argument is the same as in Proposition 5.1. The string returned by *sage* to
`R=RootSystem(["E",6,1]).weight_lattice(), Lambda=R.fundamental_weights(),`
`omega6=Lambda[6]-Lambda[0], R.reduced_word_of_translation(3*omega6)`
reads

$$\begin{aligned} & [0, 2, 4, 3, 1, 5, 4, 2, 0, 3, 4, 2, 5, 4, 3, 1, 6, 5, 4, 2, 0, 3, 4, 2, \\ & 5, 4, 3, 1, 6, 5, 4, 2, 0, 3, 4, 2, 5, 4, 3, 1, 6, 5, 4, 2, 3, 4, 5, 6]. \end{aligned}$$

Verification of the statement $\phi^6 = 2\omega_7$ in the proof of Proposition 5.2.

The argument is the same as in Proposition 5.1. The string returned by *sage* to
`R=RootSystem(["E",7,1]).weight_lattice(), Lambda=R.fundamental_weights(),`
`omega7=Lambda[7]-Lambda[0], R.reduced_word_of_translation(2*omega7)`
reads

$$(38) \quad \begin{aligned} & [0, 1, 3, 4, 2, 5, 4, 3, 1, 0, 6, 5, 4, 2, 3, 1, 4, 3, 5, 4, 2, 6, 5, 4, 3, 1, 0, \\ & 7, 6, 5, 4, 2, 3, 1, 4, 3, 5, 4, 2, 6, 5, 4, 3, 1, 7, 6, 5, 4, 2, 3, 4, 5, 6, 7]. \end{aligned}$$

References

- [1] *N. Bourbaki*, Groupes et Algèbres de Lie, ch. **4, 5, 6** / ch. **7, 8, 9**, Springer-Verlag Berlin Heidelberg (2006).
- [2] *E. Grosse-Klönne*, From pro- p -Iwahori Hecke modules to (Φ, Γ) -modules, I, Duke Math J. 165, no. 8 (2016), 1529-1595

- [3] *X. He and S. Nie*, Minimal length elements of extended affine Weyl groups, Compos. Math. **150** (2014), no. 11, 1903–1927
- [4] *N. Iwahori and H. Matsumoto*, On some Bruhat decomposition and the structure of the Hecke rings of p -adic Chevalley groups, Publ. Math. Inst. Hautes Études Sci. **25** (1965), 5–48.
- [5] *J. C. Jantzen*, Representations of Algebraic Groups, Mathematical Surveys and Monographs **107**, 2nd ed., American Mathematical Society (2003)
- [6] *K. Koziol*, Pro- p -Iwahori invariants for SL_2 and L -packets of Hecke modules, Int. Math. Res. Not. IMRN (2016), no. 4, 1090–1125.
- [7] *R. Ollivier*, Compatibility between Satake and Bernstein-type isomorphisms in characteristic p , Algebra Number Theory **5** (2014) 1071–1111.
- [8] *M.F. Vignéras*, Pro- p -Iwahori Hecke ring and supersingular $\overline{\mathbb{F}}_p$ -representations, Math. Ann. **331** (2005), no. 3, 523–556 and Math. Ann. **333** (2005), no. 3, 699–701.
- [9] *M.F. Vignéras*, The pro- p -Iwahori Hecke algebra of a reductive p -adic group III, J. Inst. Math. Jussieu (to appear).

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